Areas & Volumes

Using Double & Triple Integrals — Cartesian Coordinates Only

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Cartesian Coordinates Only!

Key Formulas: Cartesian Double & Triple Integrals

Double Integral — Area under region

$$\iint_R dA, \qquad dA = dx\,dy$$

Iterate: outer a to b, inner $g_1(x)$ to $g_2(x)$.

Triple Integral — Volume of 3D Solid

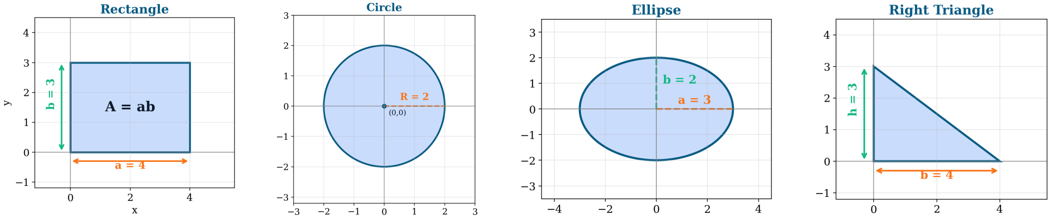
$$\iiint_E dV, \qquad dV = dx\,dy\,dz$$

Iterate limits outermost to innermost.

Strategy for Setting Up Integrals

1. **Sketch** the region — identify bounding surfaces and intersections.
2. **Choose order** — pick $dz\,dy\,dx$ etc. for simplest limits.
3. **Find limits** — innermost: function of outer vars; outermost: constants.
4. **Evaluate** step by step — compute inner integral first.

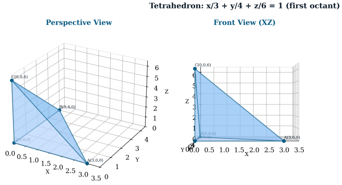
Summary: 2D Figures — Areas via Double Integrals

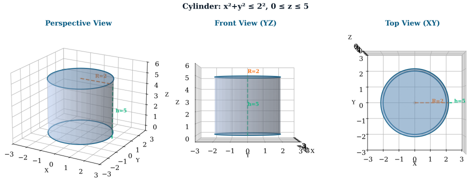


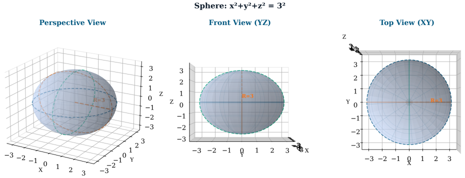
Summary: 2D Figures — Areas via Double Integrals

Figure	**Cartesian Bounds**	**Area**
Rectangle	$0 \leq x \leq a,\ 0 \leq y \leq b$	$A = ab$
Circle	$-R \leq x \leq R,\ -\sqrt{R^2 - x^2} \leq y \leq \sqrt{R^2 - x^2}$	$A = \pi R^2$
Ellipse	$-a \leq x \leq a,\ -b\sqrt{1 - x^2/a^2} \leq y \leq b\sqrt{1 - x^2/a^2}$	$A = \pi ab$
Triangle	$0 \leq x \leq b,\ 0 \leq y \leq \frac{h}{b}x$	$A = \frac{bh}{2}$

Summary: 3D Figures — Volumes via Triple Integrals





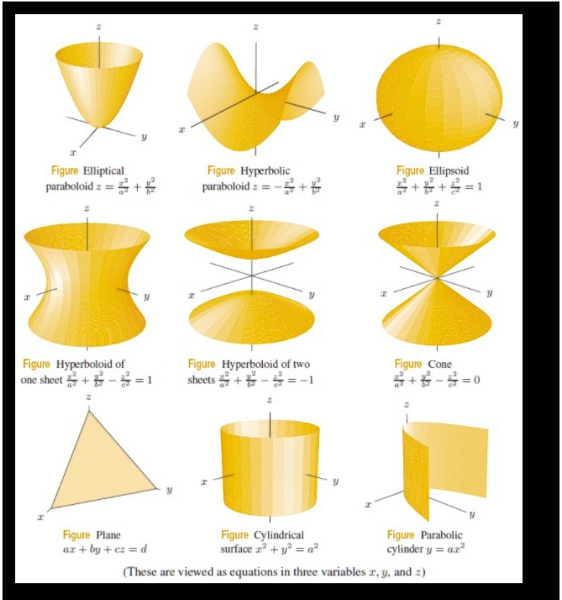


Summary: 3D Figures — Volumes via Triple Integrals

Figure	Cartesian Setup	Volume
Tetrahedron	$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$ (first octant)	$V = \frac{abc}{6}$
Cylinder	$x^2 + y^2 \leq R^2,\ 0 \leq z \leq h$	$V = \pi R^2 h$
Sphere	$x^2 + y^2 + z^2 \leq R^2$	$V = \frac{4}{3}\pi R^3$
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$	$V = \frac{4}{3}\pi abc$
Paraboloid	$z = h - x^2 - y^2,\ z \geq 0$	$V = \frac{\pi h^2}{2}$

All solved using Cartesian coordinates ($dx\,dy\,dz$) only!

Summary: 3D Figures — Volumes via Triple Integrals



Problem 1: Volume of a Tetrahedron

Problem

Find the volume of the tetrahedron bounded by $\frac{x}{3} + \frac{y}{4} + \frac{z}{6} = 1$ and the coordinate planes $(a = 3,\ b = 4,\ c = 6)$.

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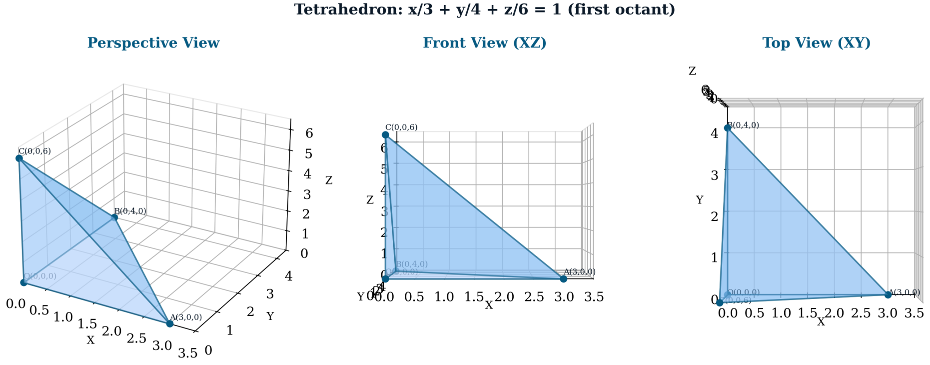

Problem 1: Solution — Volume of a Tetrahedron

Step 1 (Region): First octant, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1$. For fixed x: $0 \leq y \leq b\left(1 - \frac{x}{a}\right)$. For fixed x, y: $0 \leq z \leq c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$.

Step 2 (Setup):

$$V = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\,dy\,dx$$

Step 3 (Inner dz): $\displaystyle\int_0^{c(1-x/a-y/b)} dz = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

Step 4 (Middle dy): $\displaystyle\int_0^{b(1-x/a)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy = \frac{bc}{2}\left(1 - \frac{x}{a}\right)^2$

Step 5 (Outer dx): Let $u = 1 - \frac{x}{a}$: $\displaystyle V = \int_0^a \frac{bc}{2}\left(1 - \frac{x}{a}\right)^2 dx = \frac{abc}{6} = \frac{3 \times 4 \times 6}{6} = \boxed{12}$

Problem 2: Volume of Cylinder Between Two Planes

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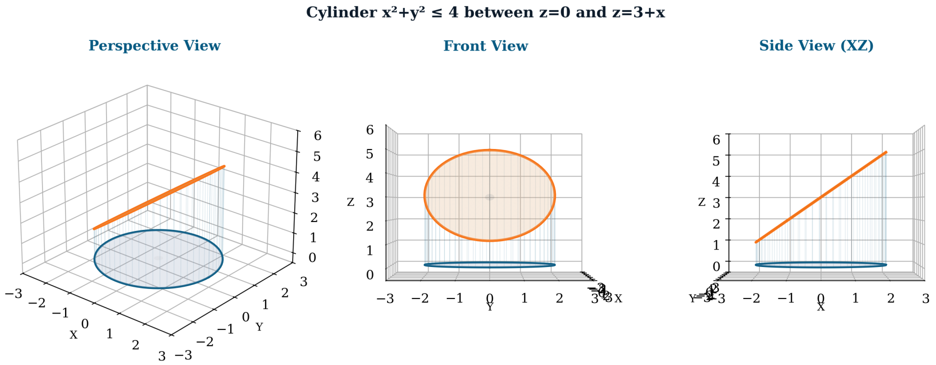

Problem 2: Solution — Cylinder Between Two Planes

Step 1 (Region): For each (x, y) in disk $x^2 + y^2 \leq 4$, z ranges from 0 to $3 + x$.

Step 2 (Setup):

$$V = \iint_{\text{disk}} \int_0^{3+x} dz\, dA$$

Step 3 (Inner dz): $\displaystyle\int_0^{3+x} dz = 3 + x$, so $V = \iint_{\text{disk}} (3 + x)\, dA$.

Step 4 (Split):

$$V = 3 \iint_{\text{disk}} dA + \iint_{\text{disk}} x\, dA$$

- First integral: $3 \cdot \pi R^2 = 3 \cdot 4\pi = 12\pi$.
- Second integral: $\iint_{\text{disk}} x\, dA = 0 \quad$ (x is an odd function over a symmetric disk).

Step 5 (Result):

$$V = 12\pi + 0 = \boxed{12\pi \approx 37.70}$$

Problem 3: Volume of a Sphere (Cartesian)

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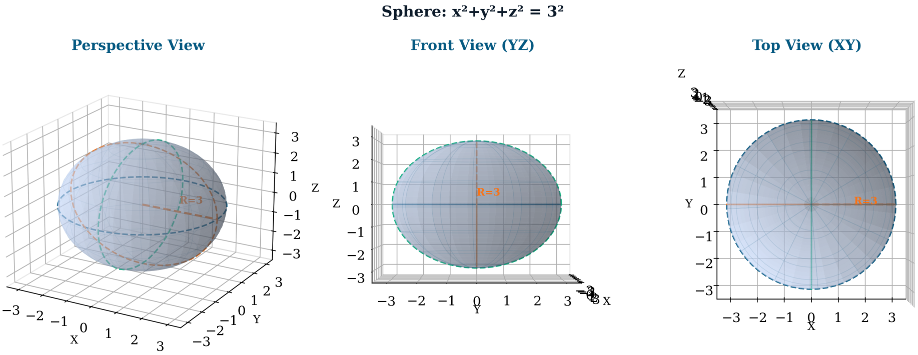

Problem 3: Solution — Volume of a Sphere

Step 1 (Limits): $-R \le x \le R, \quad -\sqrt{R^2 - x^2} \le y \le \sqrt{R^2 - x^2}, \quad -\sqrt{R^2 - x^2 - y^2} \le z \le \sqrt{R^2 - x^2 - y^2}.$

Step 2 (Setup):

$$V = \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz\, dy\, dx$$

Step 3 (Inner dz): $2\sqrt{R^2 - x^2 - y^2}$ (height of sphere at (x, y)).

Step 4 (Middle dy): Let $r^2 = R^2 - x^2$.

$$\int_{-r}^{r} 2\sqrt{r^2 - y^2}\, dy = \pi r^2 = \pi(R^2 - x^2)$$

Step 5 (Outer dx):

$$V = \int_{-R}^{R} \pi(R^2 - x^2)\, dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^{R} = \boxed{\frac{4}{3}\pi R^3}$$

Problem 4: Volume of an Ellipsoid (Substitution)

Problem

Find the volume of $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} \leq 1 \ \ (a = 3,\ b = 2,\ c = 4)$ using substitution.

Problem 4: Volume of an Ellipsoid (Substitution)

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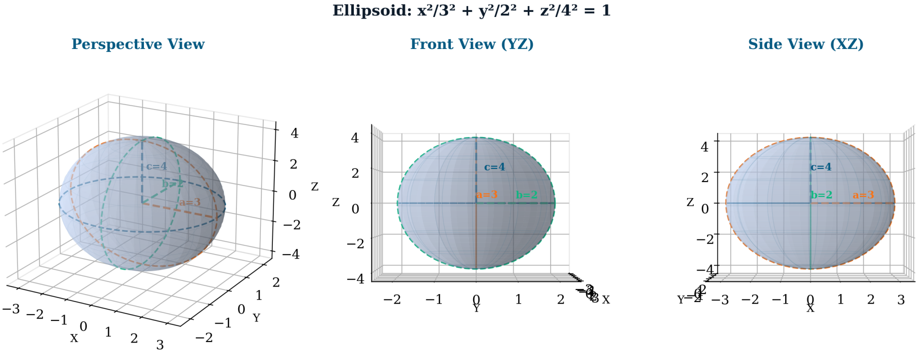

Problem 4: Solution — Volume of an Ellipsoid

Step 1 (Substitution): Let $x = 3u$, $y = 2v$, $z = 4w$. Then $u^2 + v^2 + w^2 \leq 1$ (unit sphere).

Step 2 (Setup):

$$V = \iiint_E dx\,dy\,dz = \iiint_{u^2+v^2+w^2 \leq 1} |J|\,du\,dv\,dw$$

Step 3 (Jacobian):

$$|J| = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = abc = 3 \times 2 \times 4 = 24$$

Step 4 (Evaluate):

$$V = 24 \iiint_{u^2+v^2+w^2 \leq 1} du\,dv\,dw = 24 \cdot \frac{4}{3}\pi = \boxed{32\pi \approx 100.53}$$

Problem 5: Volume Under a Paraboloid

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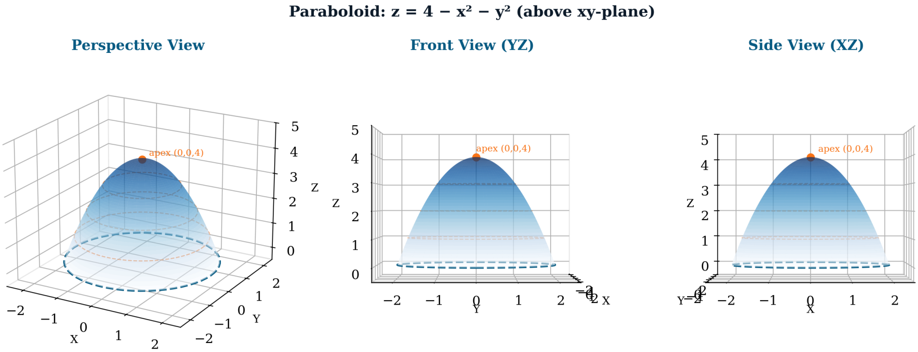

Problem 5: Solution — Volume Under a Paraboloid

Step 1 (Base): $z \geq 0 \Rightarrow x^2 + y^2 \leq 4$, a disk of radius 2.

Step 2 (Setup):

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx$$

Step 3 (Inner dy): Let $A = 4 - x^2$, limits $\pm\sqrt{A}$.

$$\int_{-\sqrt{A}}^{\sqrt{A}} (A - y^2)\, dy = \frac{4}{3} A^{3/2} = \frac{4}{3}(4 - x^2)^{3/2}$$

Step 4 (Outer dx): Substitute $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$:

$$V = \int_{-2}^{2} \frac{4}{3}(4 - x^2)^{3/2}\, dx = \frac{4}{3}\int_{-\pi/2}^{\pi/2} (4\cos^2\theta)^{3/2} \cdot 2\cos\theta\, d\theta = \frac{64}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta$$

Using $\int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{8}$:

$$V = \frac{64}{3} \cdot \frac{3\pi}{8} = \boxed{8\pi \approx 25.13}$$

Practice Exercises (1–3)

Exercise 1: Tetrahedron

Find V for the tetrahedron bounded by $x+y+z=6$, $x=0$, $y=0$, $z=0$.

Solution: Limits: $0 \le x \le 6$, $0 \le y \le 6-x$, $0 \le z \le 6-x-y$.

$$V = \int_0^6 \int_0^{6-x} \int_0^{6-x-y} dz\,dy\,dx = \int_0^6 \int_0^{6-x} (6-x-y)\,dy\,dx = \int_0^6 \frac{(6-x)^2}{2}\,dx = \boxed{36}$$

Exercise 2: Cylinder between planes

Compute V for cylinder $x^2+y^2 \le 9$, $0 \le z \le 4$.

Solution: $V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 4\,dy\,dx$. Inner: $8\sqrt{9-x^2}$. Outer uses semicircle area $= \frac{\pi(3)^2}{2}$.
$V = 4 \cdot \pi(3)^2 = \boxed{36\pi}$

Exercise 3: Hemisphere

Volume above xy-plane inside $x^2+y^2+z^2 \le 16$.

Solution: z from 0 to $\sqrt{16-x^2-y^2}$. Half of sphere volume $= \frac{1}{2} \cdot \frac{4}{3}\pi(4)^3 = \boxed{\frac{128\pi}{3}}$

Practice Exercises (4–6)

Exercise 4: Ellipsoid

Find V for $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 1$ using substitution $x = 2u,\ y = 3v,\ z = 4w$.

Solution: $|J| = 2 \cdot 3 \cdot 4 = 24$. $V = 24 \cdot \frac{4}{3}\pi = \boxed{32\pi}$

Exercise 5: Paraboloid ($h = 9$)

Find volume under $z = 9 - x^2 - y^2$ above $z = 0$.

Solution: Base: $x^2 + y^2 \leq 9$, radius 3. Same method as Problem 5 with $h = 9$:

$$V = \frac{\pi h^2}{2} = \frac{\pi \cdot 81}{2} = \boxed{\frac{81\pi}{2}}$$

Exercise 6: Between paraboloid and plane

Find volume enclosed between $z = x^2 + y^2$ and $z = 4$.

Solution: Intersection at $x^2 + y^2 = 4$. For each (x, y), z from $x^2 + y^2$ to 4:

Key Takeaways

1. **Double integrals:** $dA = dx\,dy$ for areas and volumes under surfaces.
2. **Triple integrals:** $dV = dx\,dy\,dz$ for volumes of 3D solids.
3. **Tetrahedron:** $V = \dfrac{abc}{6}$ for the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ in the first octant.
4. **Sphere:** $V = \dfrac{4}{3}\pi R^3$. Substitution for ellipsoid gives $V = \dfrac{4}{3}\pi abc$.
5. **Always sketch the region first** — it determines integration bounds!

Thank You!

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