25MT105 - Calculus and Ordinary Differential Equations

Module 1 - Pre T1 Examination

Solutions by: *Dr. D Bhanu Prakash*

Exam Date: March 09, 2026   Section: 7, 14, 21   Max Marks: 10

Paper 1 - Section 7

Question 1 [2 Marks]

Write the statement of Rolle's Theorem.

Solution

Statement of Rolle's Theorem:

Let $f : [a, b] \to \mathbb{R}$ be a function satisfying the following three conditions:

1. f is **continuous** on the closed interval $[a, b]$,

2. f is **differentiable** on the open interval (a, b), and

3. $f(a) = f(b)$.

Then there exists at least one point $c \in (a, b)$ such that:

$$\boxed{f'(c) = 0}$$

Geometric Interpretation: There exists at least one point on the curve $y = f(x)$ between $x = a$ and $x = b$ where the tangent is horizontal (parallel to the x-axis).

Question 2 [3 Marks]

Verify Lagrange's Mean Value Theorem for $f(x) = x^{1/3}$ on $[-1, 1]$.

Solution

Statement of LMVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists\, c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

1

Step 1 — Continuity:

$f(x) = x^{1/3}$ is defined and continuous for all $x \in \mathbb{R}$, hence continuous on $[-1, 1]$.

Step 2 — Differentiability:

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}},$$

At $x = 0 \in (-1, 1)$: $f'(x) \to \infty$. So f is **not differentiable at** $x = 0$.

Conclusion: LMVT hypothesis fails. The theorem is **not applicable**.

Checking the conclusion numerically:

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{1 - (-1)}{2} = 1$$

Setting $f'(c) = 1$:

$$\frac{1}{3c^{2/3}} = 1 \implies c^{2/3} = \frac{1}{3} \implies c = \pm\frac{1}{3\sqrt{3}} \approx \pm 0.192 \in (-1, 1)$$

The conclusion holds numerically, but since differentiability fails at $x = 0$, LMVT is **not formally valid** here.

Question 3   [2 Marks]

State Taylor's Theorem with Cauchy's form of remainder.

Solution

Taylor's Theorem:

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!} f''(a) + \cdots + \frac{(x - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + R_n,$$

Cauchy's Form of Remainder:

$$R_n = \frac{(1 - \theta)^{n-1}(x - a)^n}{(n - 1)!} f^{(n)}(a + \theta(x - a)), \quad 0 < \theta < 1$$

Question 4   [3 Marks]

Verify Maclaurin's theorem for $f(x) = (1 - x^2)^{5/2}$ with Lagrange's form of remainder up to 3 terms when $x = 1$.

Solution

Maclaurin's Theorem (Lagrange's Remainder):

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + R_3, \qquad R_3 = \frac{x^3}{3!}f'''(\theta x), \quad 0 < \theta < 1$$

Step 1 — Derivatives at $x = 0$:

$$f(x) = (1 - x^2)^{5/2} \implies f(0) = 1$$

$$f'(x) = -5x(1 - x^2)^{3/2} \implies f'(0) = 0$$

$$f''(x) = -5(1 - x^2)^{3/2} + 15x^2(1 - x^2)^{1/2} \implies f''(0) = -5$$

$$f'''(x) = 45x(1 - x^2)^{1/2} - \frac{15x^3}{(1 - x^2)^{1/2}}$$

Step 2 — Expansion (3 terms):

$$f(x) = 1 + 0 + \frac{x^2}{2}(-5) + R_3 = 1 - \frac{5x^2}{2} + R_3$$

Step 3 — Verification at $x = 1$:

Left side: $f(1) = (1 - 1)^{5/2} = 0$

Sum of 3 terms: $1 - \dfrac{5}{2} = -\dfrac{3}{2}$

Required remainder: $R_3 = 0 - \left(-\dfrac{3}{2}\right) = \dfrac{3}{2}$

As $\theta \to 1^-$, the term $\dfrac{15\theta^3}{(1 - \theta^2)^{1/2}} \to \infty$, confirming a valid $\theta \in (0, 1)$ exists to satisfy the identity.

Theorem verified.

— End of Paper I —

Paper 2 - Section 14, 21

Question 1   [3 Marks]

Describe the relation between Rolle's, Lagrange's, and Cauchy's Mean Value Theorems.

Solution

1. Rolle's Theorem (RT):

Conditions: f continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$.

Conclusion: $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

2. Lagrange's Mean Value Theorem (LMVT):

Conditions: f continuous on $[a, b]$, differentiable on (a, b).

Conclusion: $\exists\, c \in (a, b)$ such that:
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

RT is a special case of LMVT: When $f(a) = f(b)$, the right-hand side becomes 0, giving $f'(c) = 0$, which is exactly Rolle's Theorem.

3. Cauchy's Mean Value Theorem (CMVT):

Conditions: f, g continuous on $[a, b]$, differentiable on (a, b), $g'(x) \neq 0$.

Conclusion: $\exists\, c \in (a, b)$ such that:
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

LMVT is a special case of CMVT: Setting $g(x) = x$ gives $g'(c) = 1$ and $g(b) - g(a) = b - a$, reducing CMVT to exactly LMVT.

Hierarchy:

$$\boxed{\text{Rolle's Theorem} \;\subset\; \text{LMVT} \;\subset\; \text{CMVT}}$$

Each theorem is a generalization of the previous. Rolle's is the most restrictive; CMVT is the most general.

Question 2 [2 Marks]

Test whether LMVT applies to $f(x) = x^{2/3}$ on $[-1, 1]$.

Solution

Given: $f(x) = x^{2/3}, \quad [a, b] = [-1, 1]$.

Condition 1 — Continuity:

$f(x) = x^{2/3}$ is continuous for all $x \in \mathbb{R}$, hence continuous on $[-1, 1]$.

Condition 2 — Differentiability:

$$f'(x) = \frac{2}{3}\, x^{-1/3} = \frac{2}{3x^{1/3}}$$

At $x = 0 \in (-1, 1)$: $f'(x) \to \infty$. So f is **not differentiable at** $x = 0$. $\times$

Conclusion: LMVT is **not applicable** to $f(x) = x^{2/3}$ on $[-1, 1]$ since differentiability fails at $x = 0$.

Remark: Even the conclusion fails here. Formally solving:

$$\frac{f(1) - f(-1)}{2} = \frac{1 - 1}{2} = 0 \implies f'(c) = \frac{2}{3c^{1/3}} = 0$$

which has **no solution** for any c. This confirms both the hypothesis and conclusion of LMVT fail.

Question 3 [2 Marks]

Using Taylor's theorem, express $f(x) = 2x^3 + 7x^2 + x - 6$ in powers of $(x - 1)$.

Solution

Expanding $f(x)$ about $a = 1$ using Taylor's theorem:

$$f(x) = \sum_{k=0}^{3} \frac{(x - 1)^k}{k!} f^{(k)}(1)$$

Step 1 — Compute derivatives at $x = 1$:

$$f(x) = 2x^3 + 7x^2 + x - 6 \implies f(1) = 2 + 7 + 1 - 6 = \mathbf{4}$$

$$f'(x) = 6x^2 + 14x + 1 \implies f'(1) = 6 + 14 + 1 = \mathbf{21}$$

$$f''(x) = 12x + 14 \implies f''(1) = 12 + 14 = \mathbf{26}$$

$$f'''(x) = 12 \implies f'''(1) = \mathbf{12}$$

Step 2 — Taylor expansion:

$$f(x) = 4 + 21(x - 1) + \frac{26}{2}(x - 1)^2 + \frac{12}{6}(x - 1)^3$$

$$\boxed{f(x) = 4 + 21(x - 1) + 13(x - 1)^2 + 2(x - 1)^3}$$

Verification at $x = 2$:

Original: $f(2) = 2(8) + 7(4) + 2 - 6 = 16 + 28 + 2 - 6 = 40$

Expansion: $4 + 21(1) + 13(1) + 2(1) = 4 + 21 + 13 + 2 = 40$

Question 4 [3 Marks]

Verify Maclaurin's theorem for $f(x) = (1 - x)^{5/2}$ with Lagrange's form of remainder up to 3 terms when $x = 1$.

Solution

Maclaurin's Theorem (Lagrange's Remainder, 3 terms):

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + R_3, \qquad R_3 = \frac{x^3}{3!}f'''(\theta x), \quad 0 < \theta < 1$$

Step 1 — Derivatives at $x = 0$:

$$f(x) = (1-x)^{5/2} \implies f(0) = 1$$

$$f'(x) = -\frac{5}{2}(1-x)^{3/2} \implies f'(0) = -\frac{5}{2}$$

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \implies f''(0) = \frac{15}{4}$$

$$f'''(x) = -\frac{15}{8}(1-x)^{-1/2} \implies f'''(\theta x) = -\frac{15}{8}(1-\theta x)^{-1/2}$$

Step 2 — Expansion:

$$f(x) = 1 - \frac{5x}{2} + \frac{15x^2}{8} + R_3$$

Step 3 — Remainder:

$$R_3 = \frac{x^3}{6} \cdot \left(-\frac{15}{8(1-\theta x)^{1/2}}\right) = -\frac{5x^3}{16(1-\theta x)^{1/2}}$$

Step 4 — Verification at $x = 1$:

Left side: $f(1) = (1-1)^{5/2} = 0$

Sum of 3 terms: $1 - \dfrac{5}{2} + \dfrac{15}{8} = \dfrac{8 - 20 + 15}{8} = \dfrac{3}{8}$

Required remainder: $R_3 = 0 - \dfrac{3}{8} = -\dfrac{3}{8}$

Solving for θ:

$$-\frac{5}{16(1-\theta)^{1/2}} = -\frac{3}{8} \implies (1-\theta)^{1/2} = \frac{5}{6} \implies 1-\theta = \frac{25}{36} \implies \theta = \frac{11}{36} \approx 0.306$$

Since $\theta = \dfrac{11}{36} \in (0,1)$, **Maclaurin's theorem is verified.**

— End of Paper II —