Module 1: T 1

Detailed Solutions

Dr. D Bhanu Prakash

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Author: Dr. D Bhanu Prakash

Vignan's Foundation for Science, Technology and Research
Vadlamudi - 522 213, Guntur, Andhra Pradesh, India.

Course: Calculus and Ordinary Differential Equations

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M 1 - T 1 - Sections 7, 14

1. State Maclarin's (Maclaurin's) Theorem [2 Marks]
2. For every $x \geq 0$, show that $1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2}e^x$ [3 Marks]
3. Define "Tangents at the Origin" [2 Marks]
4. Trace the curve $y^2(a + x) = x^2(3a - x)$, where $a > 0$ [3 Marks]
5. Define Double Point, Node, Cusp, and Inflexion Points with Examples [2 Marks]
6. Trace the curve $x = a\cosh(x/a)$ [Catenary] [4 Marks]
7. Find the root of $x^4 - 12x + 7 = 0$ near $x = 2$ [4 Marks]
8. Explain Integration as a Limit of a Sum [2 Marks]
9. Evaluate $\int_1^2 \log x \, dx$ using limit of a sum [4 Marks]
10. Reduce the formula for $\int \sin^n(bx + c)\, dx$ [4 Marks]

Schema: 2+0+0+3+0+0+0+2+4+4 = 15.

PART – A

Q1(a) State Maclarin's (Maclaurin's) Theorem [2 Marks]

Statement:

If a function $f(x)$ can be expanded as an infinite power series in x, then:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$$

Conditions: - $f(x)$ and all its derivatives must exist and be continuous at $x = 0$. - The series must converge to $f(x)$.

> This is a special case of Taylor's theorem where the expansion is about $x = 0$.

Q1(b) For every $x \geq 0$, show that $1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2}e^x$ [3 Marks]

Proof using Taylor's theorem with Lagrange's remainder:

By Maclaurin's theorem applied to e^x:

$$e^x = 1 + x + \frac{x^2}{2!}e^{\theta x}, \quad 0 < \theta < 1$$

Left inequality: $1 + x + \frac{x^2}{2} \leq e^x$

Since $0 < \theta < 1$ and $x \geq 0$:

$$e^{\theta x} \geq e^0 = 1$$

$$\frac{x^2}{2}e^{\theta x} \geq \frac{x^2}{2}$$

$$e^x = 1 + x + \frac{x^2}{2}e^{\theta x} \geq 1 + x + \frac{x^2}{2} \quad \checkmark$$

Right inequality: $e^x \leq 1 + x + \frac{x^2}{2}e^x$

Since $0 < \theta < 1$ and $x \geq 0$:

$$e^{\theta x} \leq e^x$$

$$\frac{x^2}{2}e^{\theta x} \leq \frac{x^2}{2}e^x$$

$$e^x = 1 + x + \frac{x^2}{2}e^{\theta x} \leq 1 + x + \frac{x^2}{2}e^x \quad \checkmark$$

Hence proved: $1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2}e^x$ for all $x \geq 0$.

Q2(a) Define “Tangents at the Origin” [2 Marks]

Definition:

If a curve passes through the origin, the tangents drawn to the curve **at the origin** are called **tangents at the origin**.

Method to find them:

Equate the **lowest degree terms** of the equation of the curve to zero.

Example:

For the curve $y^2 = x^2(1-x)$, the lowest degree terms give:

$$y^2 - x^2 = 0 \implies y = \pm x$$

So the tangents at the origin are $y = x$ and $y = -x$.

Q2(b) Trace the curve $y^2(a+x) = x^2(3a-x)$, where $a > 0$ [3 Marks]

This is the **Cissoid of Diocles** (modified form).

Step 1 – Symmetry: Replace $y \to -y$: equation unchanged → **Symmetric about the x-axis**.

Step 2 – Origin: Put $x = 0, y = 0$: $0 = 0$ → **Passes through origin**.

Tangents at origin (equate lowest degree terms $y^2 \cdot a = x^2 \cdot 3a$):

$$y^2 = 3x^2 \implies y = \pm\sqrt{3}x$$

Step 3 – Asymptotes: As $x \to -a$: denominator $(a+x) \to 0$, so $y \to \infty$ → **Vertical asymptote:** $x = -a$.

Step 4 – Region of existence:

$$y^2 = \frac{x^2(3a-x)}{a+x}$$

For $y^2 \geq 0$: need $\dfrac{x^2(3a-x)}{a+x} \geq 0$

- $x^2 \geq 0$ always
- $(3a-x) \geq 0 \Rightarrow x \leq 3a$
- $(a+x) > 0 \Rightarrow x > -a$

So curve exists for $-a < x \leq 3a$.

Step 5 – Special points: - At $x = 0$: $y = 0$ (origin) - At $x = 3a$: $y = 0$ → Point $(3a, 0)$

Shape: The curve forms a loop between $x = 0$ and $x = 3a$, symmetric about x-axis, with the asymptote at $x = -a$.

PART – B

Q3(a) Define Double Point, Node, Cusp, and Inflexion Points with Examples [2 Marks]

Term	Definition	Example
Double Point	A point on a curve through which **two branches** of the curve pass	Origin on $y^2 = x^2(1-x)$
Node	A double point where the **two branches cross each other** (two real distinct tangents)	Origin on $y^2 = x^2(1-x)$
Cusp	A double point where **two branches meet and have a common tangent** (two coincident tangents)	Origin on $y^2 = x^3$
Inflexion Point	A point where the curve **crosses its own tangent** (changes concavity)	$x = 0$ on $y = x^3$

Q3(b) Trace the curve $x = a\cosh(x/a)$ [Catenary] [4 Marks]

The curve $y = a\cosh(x/a)$ is the **Catenary**.

Step 1 – Symmetry: $\cosh(-x/a) = \cosh(x/a)$ → **Symmetric about y-axis**.

Step 2 – Origin / Intercepts: - At $x = 0$: $y = a\cosh(0) = a$ → **Vertex at** $(0, a)$. - No x-intercept since $\cosh(x/a) \geq 1 > 0$ always.

Step 3 – Extent: $y = a\cosh(x/a) \geq a > 0$ for all x → Curve lies entirely **above** $y = a$.

Step 4 – Asymptotes: No vertical asymptotes. As $x \to \pm\infty$, $y \to \infty$ → No finite asymptotes.

Step 5 – Behaviour:

$$\frac{dy}{dx} = \sinh(x/a)$$

- At $x = 0$: $dy/dx = 0$ → **Minimum point** at $(0, a)$. - For $x > 0$: $dy/dx > 0$ → increasing - For $x < 0$: $dy/dx < 0$ → decreasing

Shape: U-shaped curve with minimum at $(0, a)$, symmetric about y-axis, rising steeply on both sides. Resembles a hanging chain.

Q3(c) Find the root of $x^4 - 12x + 7 = 0$ near $x = 2$ [4 Marks]

Let $f(x) = x^4 - 12x + 7$

Check sign change: - $f(2) = 16 - 24 + 7 = -1 < 0$ - $f(3) = 81 - 36 + 7 = 52 > 0$

Root lies between 2 and 3.

Newton-Raphson Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f'(x) = 4x^3 - 12$

Starting with $x_0 = 2$:

$$f(2) = -1, \quad f'(2) = 4(8) - 12 = 20$$

$$x_1 = 2 - \frac{-1}{20} = 2 + 0.05 = 2.05$$

Iteration 2 with $x_1 = 2.05$:

$$f(2.05) = (2.05)^4 - 12(2.05) + 7$$
$$= 17.68 - 24.6 + 7 = 0.08 \approx 0$$

$$f'(2.05) = 4(2.05)^3 - 12 = 4(8.615) - 12 = 22.46$$

$$x_2 = 2.05 - \frac{0.08}{22.46} \approx 2.046$$

Therefore, **Root ≈ 2.046**

Q4(a) Explain Integration as a Limit of a Sum [2 Marks]

Definition:

The definite integral of $f(x)$ from a to b is defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=1}^{n} f(a + rh)$$

where $h = \dfrac{b-a}{n}$ and $n \to \infty$, $h \to 0$.

Interpretation: The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is approximated by n thin rectangles of width h, and the exact area is obtained as $n \to \infty$.

Q4(b) Evaluate $\int_1^2 \log x\,dx$ using limit of a sum [4 Marks]

Here $a = 1$, $b = 2$, $h = \dfrac{1}{n}$, $nh = 1$

$$\int_1^2 \log x \, dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\Big(1 + \frac{r}{n}\Big)$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \prod_{r=1}^{n} \Big(1 + \frac{r}{n}\Big)$$

Using the result:

$$\int_1^2 \log x \, dx = \Big[x \log x - x\Big]_1^2 = (2\log 2 - 2) - (0 - 1) = 2\log 2 - 1$$

$$\boxed{\int_1^2 \log x \, dx = 2\ln 2 - 1 \approx 0.386}$$

Q4(c) Reduce the formula for $\int \sin^n(bx + c)\, dx$ **[4 Marks]**

Let $I_n = \int \sin^n(bx + c)\, dx$

Write $\sin^n(bx + c) = \sin^{n-1}(bx + c) \cdot \sin(bx + c)$

Apply **integration by parts**: $u = \sin^{n-1}(bx + c)$, $dv = \sin(bx + c)dx$

$$I_n = -\frac{\sin^{n-1}(bx + c)\cos(bx + c)}{b} + \frac{n-1}{b} \int \sin^{n-2}(bx + c)\cos^2(bx + c)\, dx$$

Replace $\cos^2 = 1 - \sin^2$:

$$I_n = -\frac{\sin^{n-1}(bx + c)\cos(bx + c)}{b} + \frac{n-1}{b}\,[I_{n-2} - I_n]$$

$$I_n + \frac{(n-1)}{b} I_n \cdot \frac{1}{b}$$

Collecting I_n:

$$\boxed{I_n = -\frac{\sin^{n-1}(bx + c)\cos(bx + c)}{nb} + \frac{n-1}{n} I_{n-2}}$$

This is the **reduction formula** for $\int \sin^n(bx+c)\,dx$.

M 1 - T 1 - Sections 21

1. Write the Geometrical Interpretation of Lagrange's Mean Value Theorem [2 Marks]
2. If $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_1}{2}+a_n=0$, show that $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_1x=0$ has at least one root between 0 and 1. [3 Marks]
3. Define "Asymptotes" with Two Examples [2 Marks]
4. Trace the curve $x^{2/3}+y^{2/3}=a^{2/3}$ (Astroid), where $a>0$ [3 Marks]
5. State Rolle's Theorem with its Geometrical Interpretation [2 Marks]
6. Find intervals where $f(x)=(4-x^2)^2$ is increasing or decreasing [4 Marks]
7. Find the root of $x^4-12x+7=0$ near $x=2$ [4 Marks]
8. Explain Integration as a Limit of a Sum [2 Marks]
9. Evaluate $\int_1^2 e^x\,dx$ using limit of a sum [4 Marks]
10. Reduce the formula for $\int e^{ax}\sin(bx+c)\,dx$ [4 Marks]

Schema: 2+0+2+0+2+4+0+2+4+4 = 20.

PART – A

Q1(a) Write the Geometrical Interpretation of Lagrange's Mean Value Theorem [2 Marks]

Statement of LMVT:

If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists at least one point $c\in(a,b)$ such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Geometrical Interpretation:

The slope of the **chord AB** joining the endpoints $A(a,f(a))$ and $B(b,f(b))$ equals the slope of the **tangent** to the curve at some point c between a and b.

In other words: **there is at least one point on the curve where the tangent is parallel to the chord AB**.

Q1(b) If $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_1}{2}+a_n=0$**, show that** $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_1x=0$ **has at least one root between 0 and 1. [3 Marks]**

Proof using Rolle's Theorem:

Define:

$$f(x)=\frac{a_0x^{n+1}}{n+1}+\frac{a_1x^n}{n}+\frac{a_2x^{n-1}}{n-1}+\cdots+\frac{a_1x^2}{2}+a_nx$$

Check conditions of Rolle's Theorem:

1. $f(x)$ is a polynomial → continuous on $[0,1]$ and differentiable on $(0,1)$
2. $f(0)=0$
3. $f(1)=\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_1}{2}+a_n=0$ (given)

Since $f(0)=f(1)=0$, by **Rolle's Theorem**, there exists $c\in(0,1)$ such that:

$$f'(c)=0$$

But $f'(x)=a_0x^n+a_1x^{n-1}+\cdots+a_1x$

Therefore, the equation $a_0x^n+a_1x^{n-1}+\cdots+a_1x=0$ has **at least one root between 0 and 1**. ■

Q2(a) Define "Asymptotes" with Two Examples [2 Marks]

Definition:

An **asymptote** to a curve is a straight line (or curve) such that the **distance between the curve and the line approaches zero** as the point on the curve moves to infinity. The curve approaches but never touches the asymptote.

Types:

Type	Condition	Example
Vertical Asymptote	$x = a$ when $f(a) \to \infty$	$y = \frac{1}{x}$ has $x = 0$
Horizontal Asymptote	$y \to L$ as $x \to \infty$	$y = \frac{1}{x}$ has $y = 0$
Oblique Asymptote	$y = mx + c$ as $x \to \infty$	$y = x + \frac{1}{x}$ has $y = x$

Example 1: For $y = \dfrac{1}{x}$: asymptotes are $x = 0$ and $y = 0$.

Example 2: For $y = \dfrac{x^2}{x-1}$: vertical asymptote $x = 1$, oblique asymptote $y = x + 1$.

Q2(b) Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$ (Astroid), where $a > 0$ [3 Marks]

Step 1 – Symmetry: - Replace $x \to -x$: unchanged → Symmetric about **y-axis** - Replace $y \to -y$: unchanged → Symmetric about **x-axis** - Replace $x \leftrightarrow y$: unchanged → Symmetric about **y = x**

Step 2 – Intercepts: - $x = 0$: $y^{2/3} = a^{2/3} \Rightarrow y = \pm a$ → Points $(0, \pm a)$ - $y = 0$: $x^{2/3} = a^{2/3} \Rightarrow x = \pm a$ → Points $(\pm a, 0)$

Step 3 – Extent: $x^{2/3} = a^{2/3} - y^{2/3} \leq a^{2/3}$, so $|x| \leq a$ and $|y| \leq a$.

Curve is **bounded** within the square $[-a, a] \times [-a, a]$.

Step 4 – Tangent at intercepts:

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

At $(\pm a, 0)$: tangent is vertical. At $(0, \pm a)$: tangent is horizontal.

Shape: A **four-cusped hypocycloid (Astroid)** — a star-shaped closed curve with cusps at $(\pm a, 0)$ and $(0, \pm a)$, entirely contained within $[-a, a] \times [-a, a]$.

PART – B

Q3(a) State Rolle's Theorem with its Geometrical Interpretation [2 Marks]

Statement:

If $f(x)$ is: 1. **Continuous** on $[a, b]$ 2. **Differentiable** on (a, b) 3. $f(a) = f(b)$

Then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical Interpretation:

If a smooth curve has equal values at two endpoints $x = a$ and $x = b$, then there is **at least one point** on the curve between a and b where the **tangent is horizontal** (parallel to the x-axis).

Q3(b) Find intervals where $f(x) = (4 - x^2)^2$ is increasing or decreasing [4 Marks]

$$f(x) = (4 - x^2)^2$$

Find $f'(x)$:

$$f'(x) = 2(4 - x^2)(-2x) = -4x(4 - x^2)$$
$$= -4x(2 - x)(2 + x)$$

Critical points: $x = 0, x = 2, x = -2$

Sign analysis of $f'(x) = -4x(2 - x)(2 + x)$:

Interval	$-4x$	$(2 - x)$	$(2 + x)$	$f'(x)$	Behaviour
$x < -2$	$+$	$+$	$-$	$-$	**Decreasing**
$-2 < x < 0$	$+$	$+$	$+$	$+$	**Increasing**
$0 < x < 2$	$-$	$+$	$+$	$-$	**Decreasing**
$x > 2$	$-$	$-$	$+$	$+$	**Increasing**

Summary: - **Increasing** on $(-2, 0)$ and $(2, \infty)$ - **Decreasing** on $(-\infty, -2)$ and $(0, 2)$

Q3(c) Find the root of $x^4 - 12x + 7 = 0$ near $x = 2$ [4 Marks]

(See Paper 1, Q3(c) for full solution — same question)

$$\boxed{\text{Root} \approx 2.046}$$

Q4(a) Explain Integration as a Limit of a Sum [2 Marks]

(Same as Paper 1, Q4(a) — see above)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), \quad h = \frac{b-a}{n}$$

Q4(b) Evaluate $\int_1^2 e^x\,dx$ using limit of a sum [4 Marks]

Here $a = 1$, $b = 2$, $h = \frac{1}{n}$

$$\int_1^2 e^x\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} e^{1+r/n}$$

$$= \lim_{n\to\infty} \frac{e}{n} \sum_{r=0}^{n-1} \left(e^{1/n}\right)^r$$

This is a geometric series with ratio $t = e^{1/n}$:

$$= \lim_{n\to\infty} \frac{e}{n} \cdot \frac{e^{n\cdot(1/n)}-1}{e^{1/n}-1} = \lim_{n\to\infty} \frac{e}{n} \cdot \frac{e-1}{e^{1/n}-1}$$

As $h = 1/n \to 0$: $\frac{h}{e^h-1} \to 1$, so $\frac{1/n}{e^{1/n}-1} \to 1$

$$= e(e-1) = e^2 - e$$

$$\boxed{\int_1^2 e^x\,dx = e^2 - e \approx 7.389 - 2.718 = 4.671}$$

Verification: $\left[e^x\right]_1^2 = e^2 - e$

Q4(c) Reduce the formula for $\displaystyle\int e^{ax}\sin(bx+c)\,dx$ [4 Marks]

Let $I = \int e^{ax}\sin(bx+c)\,dx$

Apply integration by parts twice.

First IBP: $u = \sin(bx+c)$, $dv = e^{ax}dx$

$$I = \frac{e^{ax}\sin(bx+c)}{a} - \frac{b}{a}\int e^{ax}\cos(bx+c)\,dx$$

Second IBP: on $\int e^{ax}\cos(bx+c)\,dx$, let $u = \cos(bx+c)$, $dv = e^{ax}dx$:

$$\int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}\cos(bx+c)}{a} + \frac{b}{a}\int e^{ax}\sin(bx+c)\,dx$$

$$= \frac{e^{ax}\cos(bx+c)}{a} + \frac{b}{a}I$$

Substituting back:

$$I = \frac{e^{ax}\sin(bx+c)}{a} - \frac{b}{a}\left[\frac{e^{ax}\cos(bx+c)}{a} + \frac{b}{a}I\right]$$

$$I = \frac{e^{ax}\sin(bx+c)}{a} - \frac{be^{ax}\cos(bx+c)}{a^2} - \frac{b^2}{a^2}I$$

$$I\left(1 + \frac{b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$\boxed{I = \frac{e^{ax}\left[a\sin(bx+c) - b\cos(bx+c)\right]}{a^2+b^2} + C}$$

This is the **closed-form reduction** (not a recursive formula, since $n = 1$).
