Module 2 – T1

Part B: Detailed Solutions

Dr. D Bhanu Prakash

Sections: 7, 14 (CSE) — 21 (ECE, ECEVLSI)

Question 1 (10 Marks)

(a) [BTL-3].

Evaluate $\int_0^2 \int_x^2 e^{y^2}\, dy\, dx$ by changing the order of integration.

Solution.

Step 1 – Identify the original region.

The limits are $0 \le x \le 2$ and $x \le y \le 2$. In the xy-plane this is the triangular region

$$R = \{(x, y) : 0 \le x \le 2,\ x \le y \le 2\}.$$

Equivalently, for a fixed y (ranging 0 to 2), x runs from 0 to y.

Step 2 – Reverse the order.

$$\int_0^2 \int_x^2 e^{y^2}\, dy\, dx = \int_0^2 \int_0^y e^{y^2}\, dx\, dy.$$

Step 3 – Evaluate the inner integral.

$$\int_0^y e^{y^2}\, dx = e^{y^2} \cdot x\Big|_0^y = y\, e^{y^2}.$$

Step 4 – Evaluate the outer integral.

$$\int_0^2 y\, e^{y^2}\, dy.$$

Let $u = y^2$, so $du = 2y\, dy$:

$$= \frac{1}{2} \int_0^4 e^u\, du = \frac{1}{2} \left[e^u\right]_0^4 = \frac{1}{2}(e^4 - 1).$$

$$\int_0^2 \int_x^2 e^{y^2}\, dy\, dx = \frac{e^4 - 1}{2}.$$

(b) [BTL-4].

Evaluate the triple integral $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\,dz\,dy\,dx$.

Solution.

Step 1 – Innermost integral (over z, 0 to $1-x-y$):

$$\int_0^{1-x-y} xyz\,dz = xy \cdot \frac{z^2}{2}\bigg|_0^{1-x-y} = \frac{xy(1-x-y)^2}{2}.$$

Step 2 – Middle integral (over y, 0 to $1-x$). Let $a = 1-x$:

$$\int_0^a \frac{x\,y(a-y)^2}{2}\,dy = \frac{x}{2}\int_0^a y(a^2-2ay+y^2)\,dy = \frac{x}{2}\int_0^a (a^2y-2ay^2+y^3)\,dy.$$

$$= \frac{x}{2}\left[\frac{a^2y^2}{2} - \frac{2ay^3}{3} + \frac{y^4}{4}\right]_0^a = \frac{x}{2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{x}{2}\cdot a^4\left(\frac{6-8+3}{12}\right) = \frac{x\,a^4}{24}.$$

Substituting $a = 1-x$:

$$= \frac{x(1-x)^4}{24}.$$

Step 3 – Outer integral (over x, 0 to 1):

$$\int_0^1 \frac{x(1-x)^4}{24}\,dx = \frac{1}{24}\int_0^1 x(1-x)^4\,dx.$$

Using the Beta function: $\int_0^1 x^m(1-x)^n\,dx = B(m+1,n+1) = \dfrac{m!\,n!}{(m+n+1)!}$

$$\int_0^1 x(1-x)^4\,dx = B(2,5) = \frac{1!\cdot 4!}{6!} = \frac{24}{720} = \frac{1}{30}.$$

Therefore:

$$\frac{1}{24}\cdot\frac{1}{30} = \frac{1}{720}.$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\,dz\,dy\,dx = \frac{1}{720}.$$

(c) [BTL-3].

Solve the first-order ODE $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$ using an integrating factor. Interpret the long-term behaviour when x represents time and y is pollutant concentration.

Solution.

Step 1 – Identify the integrating factor.

This is a linear first-order ODE of the form $\frac{dy}{dx}+P(x)y=Q(x)$ with $P(x)=\frac{1}{x}$, $Q(x)=x^2$.

Integrating factor:

$$\mu(x)=e^{\int P(x)\,dx}=e^{\int \frac{1}{x}\,dx}=e^{\ln x}=x.$$

Step 2 – Multiply through by $\mu(x)=x$.

$$x\frac{dy}{dx}+y=x^3 \implies \frac{d}{dx}(xy)=x^3.$$

Step 3 – Integrate both sides.

$$xy=\int x^3\,dx=\frac{x^4}{4}+C.$$

Step 4 – General solution.

$$y=\frac{x^3}{4}+\frac{C}{x}.$$

Step 5 – Long-term behaviour.

As $x\to\infty$ (time increases): if $C>0$ the term $C/x\to 0$, so $y\approx x^3/4$, i.e. the pollutant concentration grows *without bound* (dominated by the source term x^2). If $C<0$ and small $|C|$, the transient C/x decays rapidly, and the steady-state concentration still grows polynomially. This means the lake is *unable to dilute the continuous pollutant input*, and remediation is needed.

$$y=\frac{x^3}{4}+\frac{C}{x}.$$

Long-term: concentration grows as $\sim x^3/4$; the lake cannot purge the continuous input.

Question 2 (10 Marks)

(a).

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x - y - z)\, dz\, dy\, dx.$

Solution.

Step 1 – Innermost integral (over z):

$$\int_0^{1-x-y} (x - y - z)\, dz = \left[(x - y)z - \tfrac{z^2}{2}\right]_0^{1-x-y}.$$

Let $s = 1 - x - y$. Then:

$$= (x - y)s - \frac{s^2}{2} = (x - y)(1 - x - y) - \frac{(1 - x - y)^2}{2}.$$

Note $x - y = 1 - (1 - x + y)$ and $1 - x - y = s$, so:

$$= s(x - y) - \frac{s^2}{2}.$$

Step 2 – Middle integral (over y, 0 to $1 - x$). Let $a = 1 - x$:

$$\int_0^a \left[(x - y)(a - y) - \frac{(a - y)^2}{2}\right] dy.$$

Let $t = a - y$ ($dt = -dy$), $y = a - t$, $x - y = x - (a - t) = x - a + t = t - (1 - x - x)$; since $a = 1 - x$, $x - a = 2x - 1$:

$$= \int_0^a \left[(2x - 1 + t)\, t - \frac{t^2}{2}\right] dt = \int_0^a \left[(2x - 1)t + t^2 - \frac{t^2}{2}\right] dt = \int_0^a \left[(2x - 1)t + \frac{t^2}{2}\right] dt.$$

$$= (2x - 1)\frac{a^2}{2} + \frac{a^3}{6} = \frac{(2x - 1)(1 - x)^2}{2} + \frac{(1 - x)^3}{6}.$$

Step 3 – Outer integral (over x, 0 to 1):

$$I = \int_0^1 \left[\frac{(2x - 1)(1 - x)^2}{2} + \frac{(1 - x)^3}{6}\right] dx.$$

Expand $(2x - 1)(1 - x)^2 = (2x - 1)(1 - 2x + x^2) = 2x - 4x^2 + 2x^3 - 1 + 2x - x^2 = 2x^3 - 5x^2 + 4x - 1$.

$$\int_0^1 \frac{2x^3 - 5x^2 + 4x - 1}{2}\, dx = \frac{1}{2}\left[\frac{x^4}{2} - \frac{5x^3}{3} + 2x^2 - x\right]_0^1 = \frac{1}{2}\left(\frac{1}{2} - \frac{5}{3} + 2 - 1\right)$$

$$= \frac{1}{2} \cdot \frac{3 - 10 + 12 - 6}{6} = \frac{1}{2} \cdot \frac{-1}{6} = -\frac{1}{12}.$$

$$\int_0^1 \frac{(1 - x)^3}{6}\, dx = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}.$$

$$I = -\frac{1}{12} + \frac{1}{24} = \frac{-2 + 1}{24} = -\frac{1}{24}.$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x - y - z)\, dz\, dy\, dx = -\frac{1}{24}.$$

(b).

A chemical decomposes at a rate proportional to its amount. Initially 500 g; after 3 hr, 400 g. A second substance grows from 20 g with growth constant $k = 0.15\,\text{hr}^{-1}$. Find when both have equal mass. Determine which model dominates long-term.

Solution.

Substance A (decay).

$$A(t) = 500\, e^{-\lambda t}.$$

Given $A(3) = 400$:

$$400 = 500\, e^{-3\lambda} \implies e^{-3\lambda} = 0.8 \implies \lambda = -\frac{\ln(0.8)}{3} = \frac{\ln(1.25)}{3} \approx \frac{0.22314}{3} \approx 0.07438\,\text{hr}^{-1}.$$

Substance B (growth).

$$B(t) = 20\, e^{0.15\, t}.$$

Finding t when $A(t) = B(t)$:

$$500\, e^{-0.07438\, t} = 20\, e^{0.15\, t} \implies \frac{500}{20} = e^{(0.15+0.07438)\, t} \implies 25 = e^{0.22438\, t}.$$

$$t = \frac{\ln 25}{0.22438} = \frac{3.21888}{0.22438} \approx \boxed{14.35 \text{ hr}}.$$

Verification:

$$A(14.35) = 500\, e^{-0.07438\times 14.35} \approx 500\times 0.346 \approx 173 \text{ g}, \quad B(14.35) = 20\, e^{0.15\times 14.35} \approx 20\times 8.65 \approx 173 \text{ g. } \checkmark$$

Long-run dominance.

As $t \to \infty$, $A(t) \to 0$ (decaying exponential) while $B(t) \to \infty$ (growing exponential). Therefore, the **growth model (Substance B) dominates** in the long run. Exponential growth always overtakes exponential decay regardless of initial magnitudes, because growth compounds while decay asymptotically approaches zero.

Both substances have equal mass at $t \approx 14.35$ hr. Long-term: Substance B (exponential growth) dominates; $A(t) \to 0$ while $B(t) \to \infty$.

(c).

Solve $y'' - 4y' + 3y = 0$.

Solution.

Step 1 – Characteristic equation.

Substituting $y = e^{rx}$ gives:

$$r^2 - 4r + 3 = 0 \implies (r-1)(r-3) = 0 \implies r_1 = 1, \quad r_2 = 3.$$

Two distinct real roots, so the general solution is:

$$y = C_1\,e^{x} + C_2\,e^{3x}, \qquad C_1, C_2 \in \mathbb{R}.$$

Question 3 (10 Marks)

(a).

Change the order of integration and evaluate $\int_0^4 \int_{\sqrt{y}}^2 \frac{x}{x^2+1}\,dx\,dy$.

Solution.

Step 1 – Original region.

The limits $0 \le y \le 4$, $\sqrt{y} \le x \le 2$ describe the region:

$$R = \{(x,y) : 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}.$$

Equivalently $y \le x^2$ and x between 0 and 2, with y from 0 up to x^2:

$$R = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le x^2\}.$$

Step 2 – Reverse the order.

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{x}{x^2+1}\,dx\,dy = \int_0^2 \int_0^{x^2} \frac{x}{x^2+1}\,dy\,dx.$$

Step 3 – Inner integral (over y):

$$\int_0^{x^2} \frac{x}{x^2+1}\,dy = \frac{x}{x^2+1} \cdot x^2 = \frac{x^3}{x^2+1}.$$

Step 4 – Outer integral (over x):

$$\int_0^2 \frac{x^3}{x^2+1}\,dx.$$

Perform polynomial division: $x^3/(x^2+1) = x - x/(x^2+1)$:

$$= \int_0^2 \left[x - \frac{x}{x^2+1}\right] dx = \left[\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right]_0^2 = \left(2 - \frac{\ln 5}{2}\right) - (0-0).$$

$$\int_0^4 \int_{\sqrt{y}}^2 \frac{x}{x^2+1}\,dx\,dy = 2 - \frac{\ln 5}{2}.$$

(b).

Solve the homogeneous ODE $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$. Classify the family of curves and describe their physical significance as isotherms.

Solution.

Step 1 – Homogeneous substitution.

The RHS is homogeneous of degree 0. Let $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{x^2 + 3v^2x^2}{2x \cdot vx} = \frac{1+3v^2}{2v}.$$

Step 2 – Separate variables.

$$x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v = \frac{1+3v^2-2v^2}{2v} = \frac{1+v^2}{2v}.$$

$$\frac{2v}{1+v^2}\,dv = \frac{dx}{x}.$$

Step 3 – Integrate.

$$\int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x} \implies \ln(1+v^2) = \ln|x| + \ln|C_1| \implies 1+v^2 = C_1 x.$$

(where $C_1 > 0$ is the constant of integration.)

Step 4 – Back-substitute $v = y/x$:

$$1 + \frac{y^2}{x^2} = C_1 x \implies \frac{x^2+y^2}{x^2} = C_1 x \implies x^2 + y^2 = C_1 x^3.$$

Writing $C = C_1$:

$$x^2 + y^2 = C\,x^3, \quad C > 0.$$

Classification. For each constant C, this is a **cubic algebraic curve** in the plane (a family of curves passing through the origin). They are *neither circles nor parabolas*, but rather a family of self-intersecting cubic curves symmetric about the x-axis.

Physical interpretation as isotherms. If y represents a spatial coordinate on a metal plate and x represents another coordinate, then curves satisfying $x^2 + y^2 = Cx^3$ are level curves of temperature (isotherms). Points along such a curve are at the same temperature. Since the curves are symmetric about the x-axis and cluster near the origin, the heat source is concentrated near $x = 0$, with temperature distribution spreading non-uniformly – characteristic of a plate with a directional thermal gradient.

(c).

Define the term *Complementary Function* and explain its role in the general solution of a non-homogeneous equation.

Solution.

Definition. Consider the nth-order linear ODE with non-constant forcing:

$$L[y] \;\equiv\; a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x), \quad f(x) \not\equiv 0.$$

The **Complementary Function (CF)** is the *general solution* of the associated homogeneous equation:

$$L[y] = 0.$$

It contains n arbitrary constants (for an nth-order ODE) and spans the null-space of the operator L.

Role in the General Solution.

1. **Particular Integral (PI):** A single *specific* function $y_p(x)$ satisfying $L[y_p] = f(x)$ (no arbitrary constants).

2. **General Solution:**

$$y_{\text{GS}} = \underbrace{y_c(x)}_{\text{CF}} + \underbrace{y_p(x)}_{\text{PI}}.$$

Why this decomposition is valid. If y_p is any particular solution and y_c satisfies $L[y_c] = 0$, then:

$$L[y_c + y_p] = L[y_c] + L[y_p] = 0 + f(x) = f(x) \quad \checkmark$$

(by linearity of L). Conversely, if y^* is *any* solution of $L[y] = f$, then $L[y^* - y_p] = 0$, so $y^* - y_p$ is in the CF. Hence $y^* = y_c + y_p$ for some y_c in the CF.

Example. For $y'' - 4y' + 3y = 6$ (non-homogeneous):

- **CF:** Solve $r^2 - 4r + 3 = 0 \Rightarrow r = 1, 3$, giving $y_c = C_1e^x + C_2e^{3x}$.
- **PI:** Try $y_p = A$ (constant); $3A = 6 \Rightarrow A = 2$.
- **General Solution:** $y = C_1e^x + C_2e^{3x} + 2$.

CF = general solution of $L[y] = 0$ (contains all n arbitrary constants).
General Solution of $L[y] = f$ = CF + PI.
The CF captures all *transient/free* behaviour; the PI captures the *driven/steady* response.

End of Module 2 – T1 Solutions