Mean Value Theorem

Advanced Problems & Complete Solutions

Dr. D Bhanu Prakash

Vignan's Foundation for Science, Technology and Research

Vadlamudi, Guntur – 522 213, Andhra Pradesh, India.

The Mean Value Theorem

Theorem (Lagrange's MVT):
If f is continuous on [a, b] and differentiable on (a, b), then there exists at least one $c \in (a, b)$ such that:

$$f(b) - f(a) = f'(c) \cdot (b - a)$$

Equivalently: $f'(c) = [f(b) - f(a)] / (b - a)$ = the average rate of change.

A powerful corollary is the Inequality Form: if $|f'(x)| \leq M$ for all $x \in (a, b)$, then $|f(b) - f(a)| \leq M|b - a|$. This is what most of the problems below exploit.

Problem 1

Problem 1: Show that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Solution

Step 1 — Set up the function: Let $f(t) = \sin t$. It is continuous and differentiable everywhere on $\mathbb{R}$.

Step 2 — Compute the derivative: $f'(t) = \cos t$, and we know $|\cos t| \leq 1$ for all $t \in \mathbb{R}$.

Step 3 — Apply the MVT: For any two real numbers x and y (WLOG $x \neq y$), the MVT on the interval [x, y] (or [y, x]) gives:

$$f(y) - f(x) = f'(c)(y - x) \quad \text{for some c between x and y.}$$

Step 4 — Take absolute values:

$$|\sin y - \sin x| = |f'(c)| \cdot |y - x| = |\cos c| \cdot |y - x| \leq 1 \cdot |y - x|$$

Conclusion: $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
This shows sin is a Lipschitz function with constant $L = 1$.

Problem 2

Problem 2: Prove that for all $x > 0$: $x / (1 + x^2) \leq \arctan x \leq x$.

Solution

Part A: $\arctan x \leq x$

Step 1 — Define f: Let $f(t) = \arctan t$. Then $f'(t) = 1/(1 + t^2)$.

Step 2 — Apply MVT on $[0, x]$ for $x > 0$:

$$\arctan x - \arctan 0 = f'(c) \cdot x \quad \text{for some } c \in (0, x).$$

$$\arctan x = x / (1 + c^2)$$

Step 3 — Bound from above: Since $c > 0$, we have $1 + c^2 > 1$, so $1/(1 + c^2) < 1$. Thus:

$$\arctan x = x / (1 + c^2) < x \quad \Rightarrow \quad \arctan x \leq x \ \checkmark$$

Part B: $\arctan x \geq x / (1 + x^2)$

Step 4 — Same MVT expression: From above, $\arctan x = x/(1 + c^2)$ for some $c \in (0, x)$.

Step 5 — Bound from below: Since $c < x$, we have $c^2 < x^2$, so $1 + c^2 < 1 + x^2$, giving $1/(1+c^2) > 1/(1+x^2)$.

$$\arctan x = x / (1 + c^2) > x / (1 + x^2) \quad \Rightarrow \quad \arctan x \geq x / (1+x^2) \ \checkmark$$

Conclusion: $x / (1 + x^2) \leq \arctan x \leq x$ for all $x > 0$.

Problem 3

Problem 3: Show that $\sinh x \geq x$ for all $x \geq 0$.

Solution

Step 1 — Define g: Let $g(x) = \sinh x - x$. We want to show $g(x) \geq 0$ for all $x \geq 0$.

Step 2 — Note g(0): $g(0) = \sinh 0 - 0 = 0$.

Step 3 — Apply the MVT on $[0, x]$ for any $x > 0$:

$$g(x) - g(0) = g'(c) \cdot x \quad \text{for some } c \in (0, x).$$
$$g(x) = g'(c) \cdot x \quad [\text{since } g(0) = 0]$$

Step 4 — Compute g': $g'(t) = \cosh t - 1$. For all $t \in \mathbb{R}$, $\cosh t = (e^t + e^{-t})/2 \geq 1$, so $g'(t) \geq 0$.

Step 5 — Conclude: Since $c > 0$ and $x > 0$, and $g'(c) = \cosh c - 1 \geq 0$:

$$g(x) = g'(c) \cdot x \geq 0 \quad \Rightarrow \quad \sinh x - x \geq 0 \quad \Rightarrow \quad \sinh x \geq x \;\checkmark$$

Conclusion: $\sinh x \geq x$ for all $x \geq 0$. Equality holds only at $x = 0$.

Note: The same argument gives $\sinh x \leq x$ for $x \leq 0$ by symmetry (sinh is odd).

Problem 4

Problem 4: Prove that $e^x \geq 1 + x + x^2/2$ for all $x \geq 0$.

Solution

Step 1 — Define h: Let $h(x) = e^x - 1 - x - x^2/2$. We need to show $h(x) \geq 0$ for $x \geq 0$.

Step 2 — Base values: $h(0) = 1 - 1 - 0 - 0 = 0$. Also $h'(x) = e^x - 1 - x$.

Step 3 — Apply MVT to h on $[0, x]$:

$$h(x) - h(0) = h'(c) \cdot x \quad \text{for some } c \in (0, x).$$
$$h(x) = (e^c - 1 - c) \cdot x$$

Step 4 — Show $h'(c) \geq 0$ by applying MVT again: Apply MVT to $\varphi(t) = e^t - 1 - t$ on $[0, c]$: $\varphi'(t) = e^t - 1$.

$$\varphi(c) - \varphi(0) = \varphi'(d) \cdot c \quad \text{for some } d \in (0, c).$$
$$e^c - 1 - c = (e^d - 1) \cdot c$$

Step 5 — Final bound: Since $d > 0$, we have $e^d > 1$, so $(e^d - 1) > 0$. Therefore $e^c - 1 - c \geq 0$, and with $x > 0$:

$$h(x) = (e^c - 1 - c) \cdot x \geq 0 \quad \Rightarrow \quad e^x \geq 1 + x + x^2/2 \;\checkmark$$

Conclusion: $e^x \geq 1 + x + x^2/2$ for all $x \geq 0$. (First three terms of the Taylor series give a lower bound.)

Problem 5

Problem 5: Show that for $0 < x < \pi/2$: $(2/\pi)x \leq \sin x \leq x$.

Solution

Part A: $\sin x \leq x$ (upper bound)

Step 1 — Apply MVT to $f(t) = \sin t$ on $[0, x]$:

$$\sin x - \sin 0 = \cos(c) \cdot x \quad \text{for some } c \in (0, x).$$
$$\sin x = x \cdot \cos c$$

Step 2 — Bound: Since $\cos c \leq 1$ for all c:

$$\sin x = x \cos c \leq x \ \checkmark$$

Part B: $\sin x \geq (2/\pi)x$ (lower bound)

Step 3 — Consider $g(t) = \sin t / t$ on $(0, \pi/2]$: We need $g(x) \geq g(\pi/2) = \sin(\pi/2) / (\pi/2) = 1/(\pi/2) = 2/\pi$.

Step 4 — Show g is decreasing: $g'(t) = [t \cos t - \sin t] / t^2$. Let $p(t) = t \cos t - \sin t$.

Step 5 — Show $p(t) \leq 0$: $p(0) = 0$ and $p'(t) = -t \sin t \leq 0$ for $t \in [0, \pi/2]$. So p is decreasing, hence $p(t) \leq p(0) = 0$.

Step 6 — Conclude: Since $g'(t) = p(t)/t^2 \leq 0$, g is non-increasing on $(0, \pi/2]$. Therefore for $x \in (0, \pi/2]$:

$$g(x) \geq g(\pi/2) = 2/\pi \quad \Rightarrow \quad \sin x / x \geq 2/\pi \quad \Rightarrow \quad \sin x \geq (2/\pi)x \ \checkmark$$

Conclusion: $(2/\pi)x \leq \sin x \leq x$ for all $x \in (0, \pi/2)$.

This is Jordan's inequality. Equality holds: left side at $x = \pi/2$, right side at $x = 0$.

Problem 6

Problem 6: Prove that $\ln(1 + x) \leq x - x^2/2(1+x)$ for $x > 0$. Equivalently: $\ln(1+x) \leq x - x^2/[2(1+x)]$.

Solution

Step 1 — Equivalently show: Define $\psi(x) = x - x^2/[2(1+x)] - \ln(1+x) \geq 0$ for $x > 0$.

Step 2 — Check $\psi(0)$: $\psi(0) = 0 - 0 - 0 = 0$.

Step 3 — Compute $\psi'(x)$:

$$\psi'(x) = 1 - [2(1+x)\cdot x - x^2\cdot 2] / [2(1+x)^2] - 1/(1+x)$$
$$= 1 - 1/(1+x) - x^2 / [2(1+x)^2] \quad \text{[after simplification]}$$

$$= x/(1+x) - x^2 / [2(1+x)^2] = x(2(1+x) - x) / [2(1+x)^2] = x(2+x) / [2(1+x)^2]$$

Step 4 — Sign of ψ': For $x > 0$: $x > 0$, $2+x > 0$, and $(1+x)^2 > 0$, so $\psi'(x) > 0$.

Step 5 — Apply MVT: By the MVT on $[0, x]$: $\psi(x) - \psi(0) = \psi'(c) \cdot x$ for some $c \in (0,x)$. Since $\psi'(c) > 0$ and $x > 0$:

$$\psi(x) = \psi'(c) \cdot x > 0 \;\Rightarrow\; \ln(1+x) \leq x - x^2/[2(1+x)] \;✓$$

Conclusion: $\ln(1+x) \leq x - x^2/[2(1+x)]$ for all $x > 0$.
This is sharper than the standard $\ln(1+x) \leq x$ bound.

Problem 7: Use the Mean Value Theorem to approximate $\sqrt{5}$, and estimate the error.

Solution

Step 1 — Choose the function: Let $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = 1 / (2\sqrt{x})$.

Step 2 — Choose the base point a: We need a nearby perfect square. Take $a = 4$ since $\sqrt{4} = 2$ is exact and 4 is close to 5.

Step 3 — Compute f(a) and f′(a):

$$f(4) = \sqrt{4} = 2$$

$$f'(4) = 1 / (2\sqrt{4}) = 1 / (2 \times 2) = 1/4 = 0.25$$

Step 4 — Apply the approximation formula with $b = 5$:

$$f(5) \approx f(4) + f'(4) \cdot (5 - 4)$$

$$\sqrt{5} \approx 2 + (1/4)(1) = 2 + 0.25 = 2.25$$

Error Analysis

Since $f(x) = \sqrt{x}$ is concave down ($f''(x) = -1/(4x^{(3/2)}) < 0$), the tangent line lies above the curve. This tells us the approximation will overestimate $\sqrt{5}$.

Accuracy Table

Value	Result	Note
MVT Approximation	2.2500	Computed above
Actual √5	2.2360679…	Calculator value
Absolute Error	0.0139…	|2.25 − 2.2361|
Relative Error	≈ 0.62%	Error / True value

Result: $\sqrt{5} \approx 2.25$ (true value: 2.2361…)

The tangent line overestimates $\sqrt{5}$ by about 0.014 (≈0.62%), confirming the concavity argument.

Why It Works

The MVT guarantees that for f continuous on $[a, b]$ and differentiable on (a, b), there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. Replacing c with a gives the linear approximation — essentially "the tangent line at a is a good local approximation to the curve." The closer b is to a, the better the approximation.

Tip: Always pick a = nearest perfect square (or perfect power) to b for the best accuracy.

Key Strategy: In all problems, choose f carefully so $|f'|$ is bounded or monotone, then apply MVT to extract the desired inequality.