Prerequisite

Differentiation

VIGNAN'S
FOUNDATION FOR SCIENCE, TECHNOLOGY & RESEARCH
(Deemed to be University) - Estd. u/s 3 of UGC Act 1956

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Limits

Limit	• If $x \to a, f(x) \to L$, then L is called limit of the function $f(x)$. • $\lim_{x\to a} f(x) = L$ • Left Hand Limit of f at $x = a$ is $\lim_{x\to a^-} f(x)$ • Right Hand Limit of f at $x = a$ is $\lim_{x\to a^+} f(x)$ • If the Right and Left Hand Limits coincide The limit of $f(x)$ at $x = a$ and $\lim_{x\to a} f(x)$		
Algebra of Limits	• $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$ • $\lim_{x\to a}[f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$ • $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ whenever $\lim_{x\to a} g(x) \neq 0$	**Limits of Trigonometric Functions**	• $\lim_{x\to 0} \frac{\sin x}{x} = 1$ • $\lim_{x\to 0} \frac{\tan x}{x} = 1$ • $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

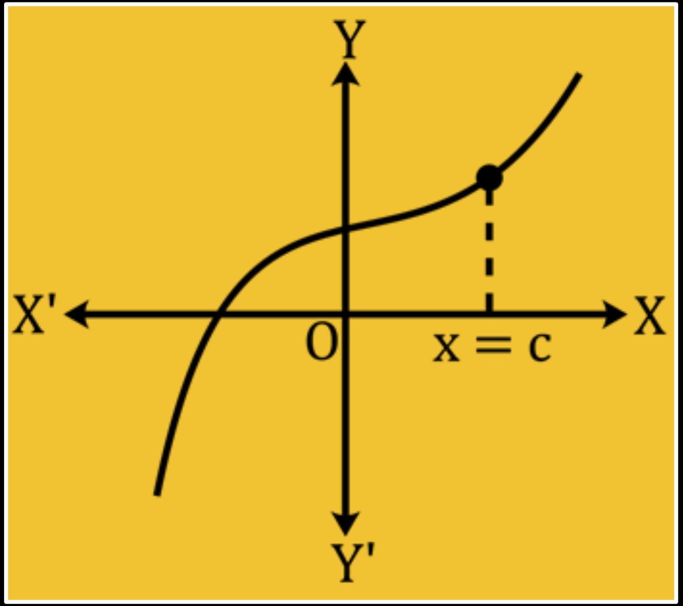
Continuity

Let **f** be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

If f is not continuous at c ⇒ f is discontinuous at c and c is called a point of discontinuity of f.



1. **Continuity of Function at point x = c**

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

1. **Geometrical Meaning of Continuity.**

In Graph → No Break or Jump → Continuous Function

In Graph → Any Break or Jump → Discontinuous

Function

Differentiability and Derivatives of Composite Functions

Example: $y = f(x)$

$$\text{Slope of CA} = \frac{f(a-h)-f(a)}{a-h-a} = \frac{f(a-h)-f(a)}{-h}$$

as $h \to 0 \Rightarrow C \to A$

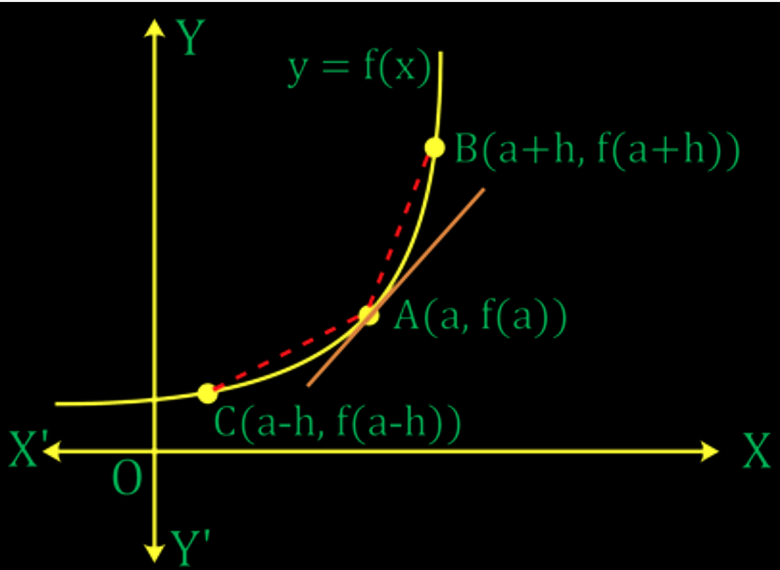
$$\text{Tangent at Point A} = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

$$\text{Slope of AB} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

as $h \to 0 \Rightarrow B \to A$

$$\text{Tangent at Point A} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$ → Left Hand Derivative (LHD)

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ → Right Hand Derivative (RHD)

Differentiability and Derivatives of Composite Functions

The derivative of a real function	f is a real function and c is a point in its domain, then the derivative of f at $x = c$ is defined by $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ provided this limit exists.
Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))\vert_c$	$f'(x) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ wherever the limit exists is defined to be the derivative of f.
The process of finding derivative of a function is called **Differentiation**.	
Differentiate $f(x)$ with respect to x to mean find $f'(x)$.	

Differentiability and Derivatives of Composite Functions

Algebra of Derivatives

1. $(u \pm v)' = u' \pm v'$
2. $(uv)' = u'v + uv'$ (Product Rule)
3. $\left(\frac{u}{v}\right)' = \frac{(u'v - uv')}{v^2}$, wherever $v \neq 0$ (Quotient Rule)

Derivatives of Some Standard Functions							
$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x \tan x$	$-\cot x \operatorname{cosec} x$

Points to Remember

If a function is

Differentiable	→	Definitely Continuous
Continuous	→	Not Necessarily Differentiable
Discontinuous	→	Definitely Not Differentiable

Summary

Differentiability of Standard Functions

All of the standard functions are differentiable except at certain points, as follows:

1. **Polynomial functions** are **differentiable** in its **domain(R).**
1. A **rational function** $\frac{p(x)}{q(x)}$ is **differentiable except where $q(x) = 0$**, where the function grows to infinity.

   E.g. $\frac{1}{x}$ and $\frac{1}{x^2}$ both functions are **not differentiable at $x = 0$**.
1. **Sines, cosines** and **exponents** are **differentiable everywhere**

   Tangents and **secants** are **not differentiable** at values where they are not defined, i.e.,

   $$x = (2n + 1)\frac{\pi}{2}, n \in Z$$

   Cotangents and **cosecants** are **not differentiable** at values where they are not defined, i.e.,

   $$x = n\pi, n \in Z$$

Summary

If the given expression is implicit function

1. Directly differentiate with respect to x
2. Separate like and unlike terms
3. Solve for **dy/dx**

Derivatives of Inverse Trigonometric Function

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{1-x^2}}$	$\text{cosec}^{-1} x$	$-\frac{1}{x\sqrt{1-x^2}}$

Exponential and Logarithmic Functions

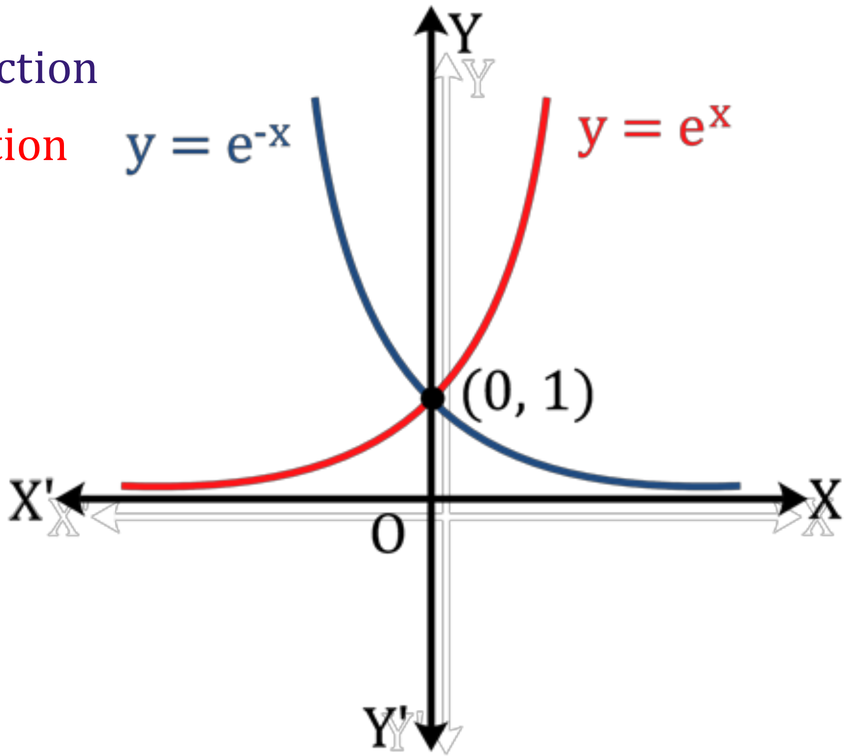
Exponential Function

$y = b^x, b > 1$

$b = 10 \Rightarrow y = 10^x \rightarrow$ Common Exponential Function

$b = e \Rightarrow y = e^x \rightarrow$ Natural Exponential Function

$e \approx 2.71828$



Nature

$y = e^x \rightarrow$ Increasing

$y = e^{-x} \rightarrow$ Decreasing

Domain : R

Range : R^+

Exponential and Logarithmic Functions

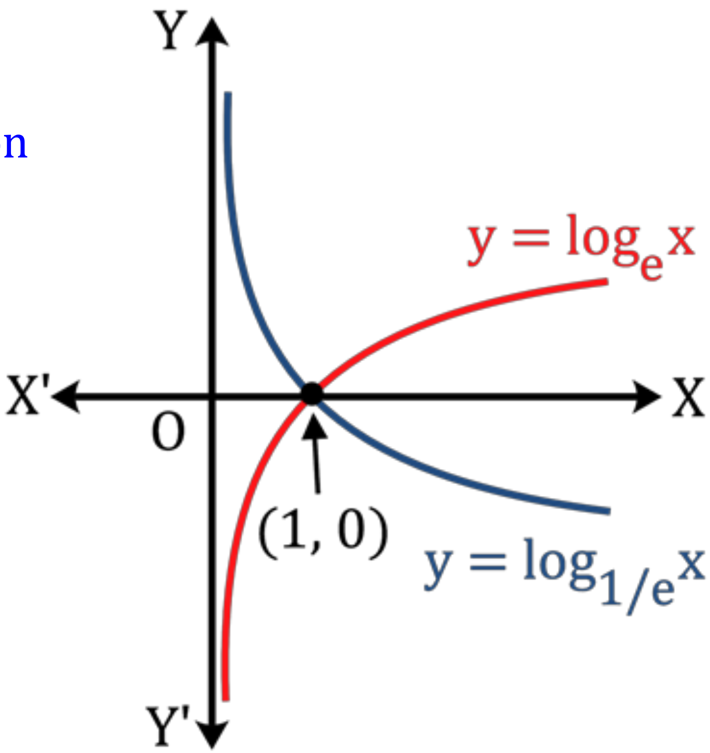
Logarithmic Function

$b^y = x \Rightarrow y = \log_b x$

$b = 10 \Rightarrow y = \log_{10} x \rightarrow$ Common Logarithmic Function

$b = e \Rightarrow y = \log_e x \rightarrow$ Natural Logarithmic Function

$\log x = \log_e x = \ln x$



Nature

$b > 1 \rightarrow$ Increasing

$0 < b < 1 \rightarrow$ Decreasing

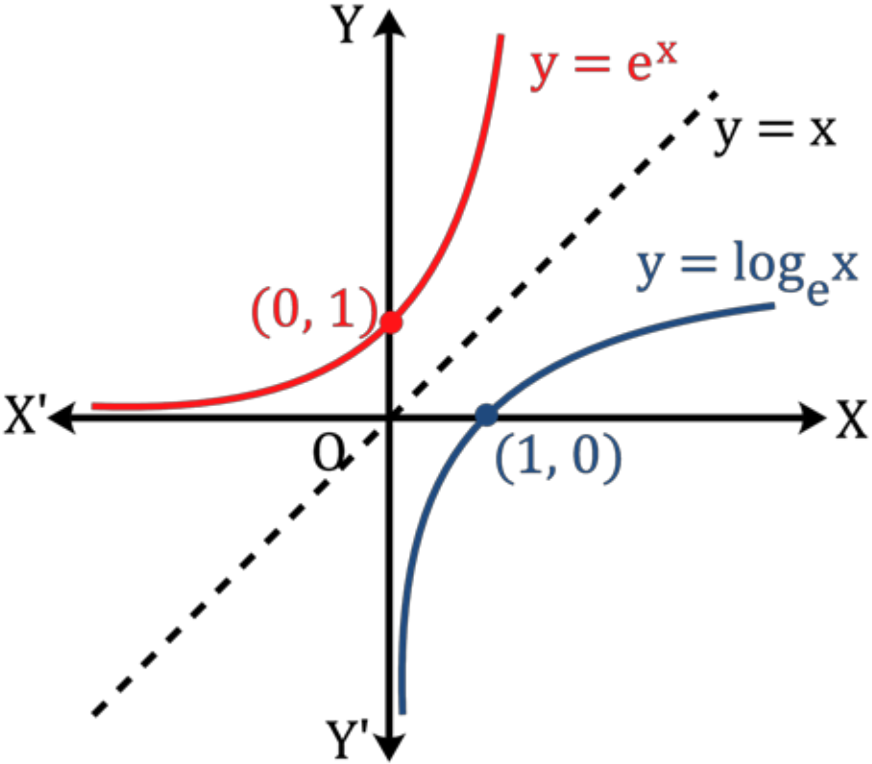
Domain : R^+

Range : R

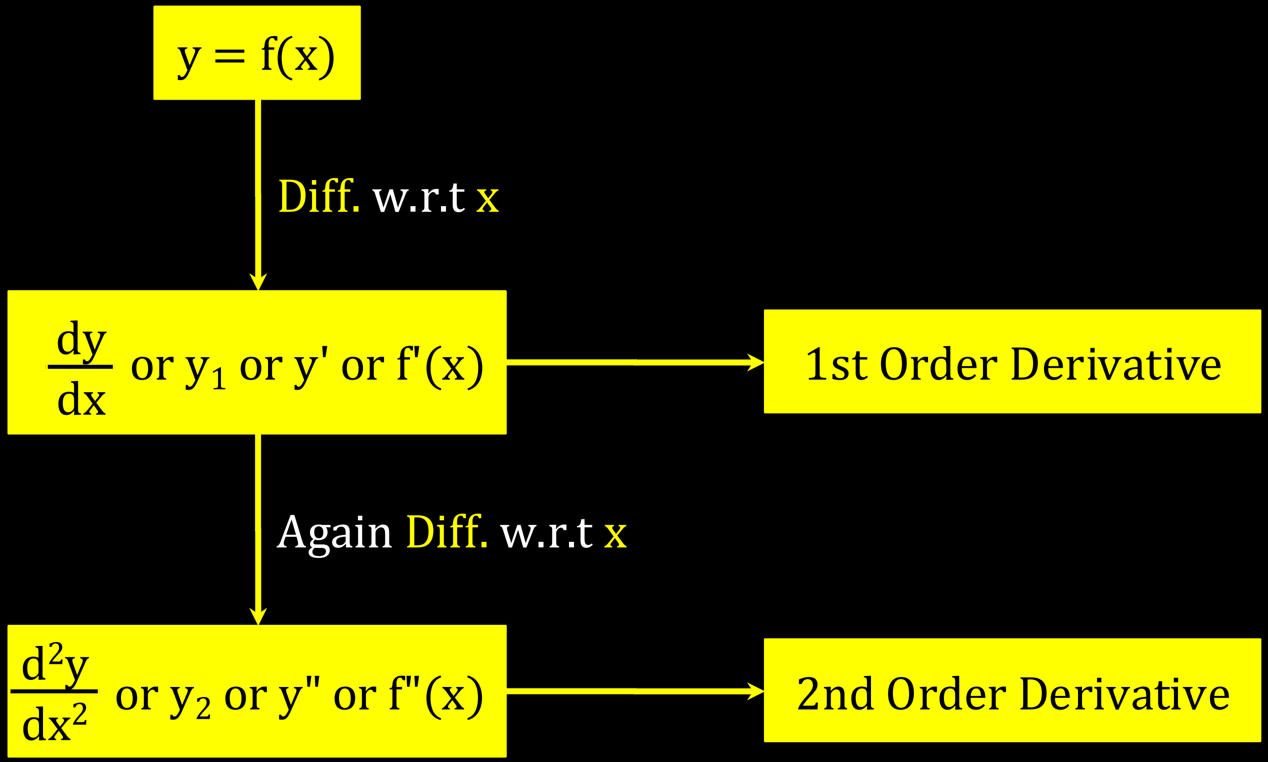
Exponential and Logarithmic Functions

Some Important Points:

1. $\log pq = \log p + \log q$
2. $\log p^n = n \log p$
1. $\log \frac{p}{q} = \log p - \log q$
1. $\log_a p = \frac{\log_e p}{\log_e a}$
1. $y = e^x$ and $y = \ln x$ are **mirror images** of each other **with respect to $y = x$.**



Second Order Derivative



Example 1 : Second Order Derivative

Example: Find $\frac{d^2y}{dx^2}$ if $y = e^x \sin 5x$.

Diff. w.r.t x

Solution: $\frac{d}{dx}(y) = \frac{d}{dx}(e^x \sin 5x) = e^x \sin 5x + 5e^x \cos 5x$

$\Rightarrow \frac{dy}{dx} = e^x(\sin 5x + 5\cos 5x)$

Again diff. w.r.t x

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^x(\sin 5x + 5\cos 5x)\right)$

Product Rule

$\Rightarrow \frac{d^2y}{dx^2} = (\sin 5x + 5\cos 5x)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\sin 5x + 5\cos 5x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^x(\sin 5x + 5\cos 5x) + e^x(5\cos 5x - 25\sin 5x)$

Simplification

$\Rightarrow \frac{d^2y}{dx^2} = 2e^x(5\cos 5x - 12\sin 5x)$

Example 2 : Second Order Derivative – Method 1

Question: If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in term of y alone.

Solution: $\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x)$

Diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Diff. w.r.t x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{d}{dx}\left((1-x^2)^{-1/2}\right)$$

Chain Rule

$$\Rightarrow \frac{d^2y}{dx^2} = -\left(-\frac{1}{2}\cdot(1-x^2)^{-3/2}\cdot -2x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{x}{(1-x^2)^{3/2}}$$

$\because y = \cos^{-1} x \Rightarrow x = \cos y$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\cos y}{(1-\cos^2 y)^{3/2}} = -\frac{\cos y}{(\sin y)^{3/2}} = -\frac{\cos y}{\sin^3 y} = -\cot y \operatorname{cosec}^2 y$$

Example 2 : Second Order Derivative – Method 2

Question: If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in term of y alone.

Solution: $\cos y = x$

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} y$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y) = -\left(-\cot y \operatorname{cosec} y \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cot y \operatorname{cosec} y \cdot (-\operatorname{cosec} y) = -\cot y \operatorname{cosec}^2 y$$

Diff. w.r.t x

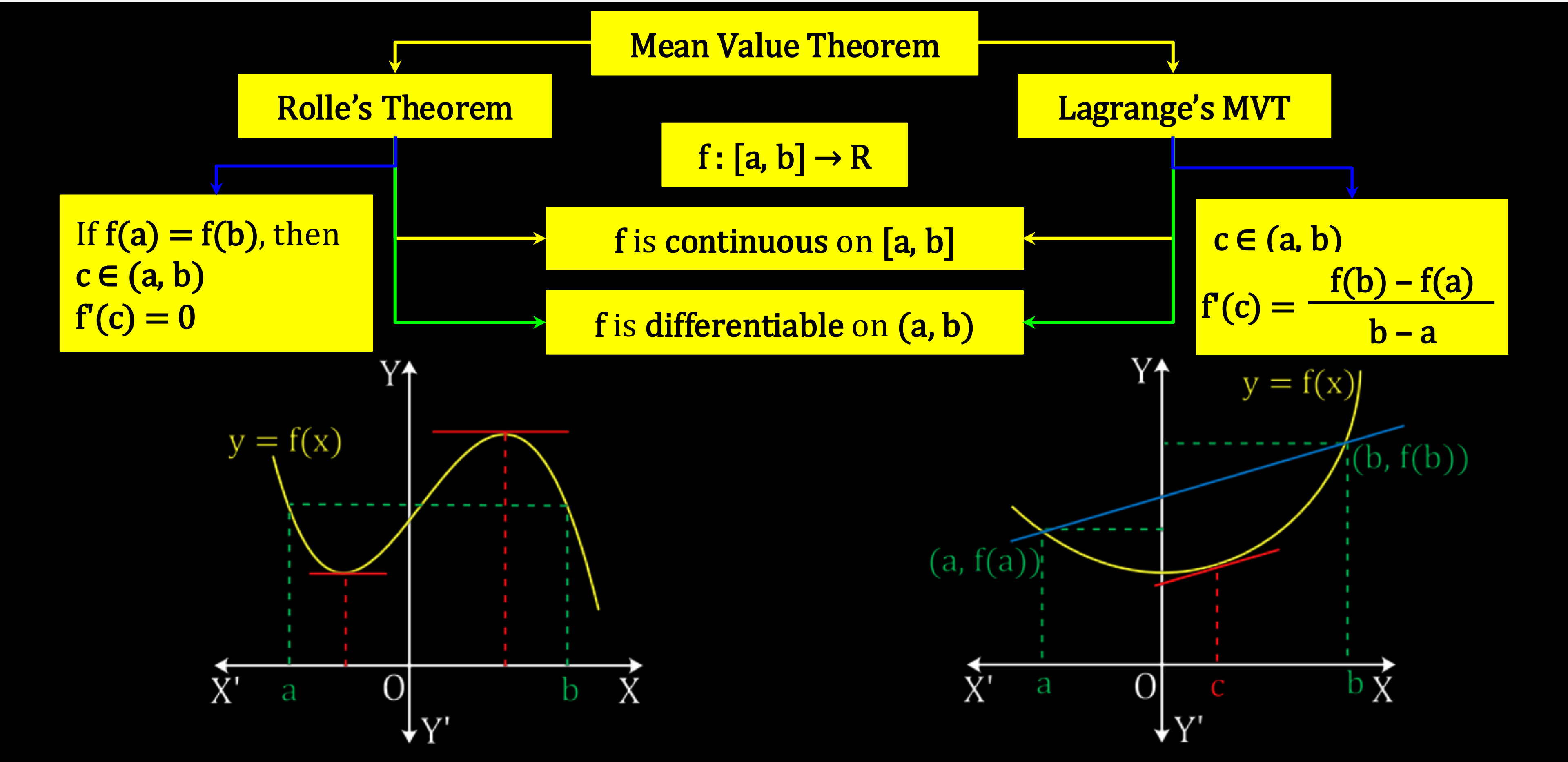
Chain Rule

Diff. w.r.t x

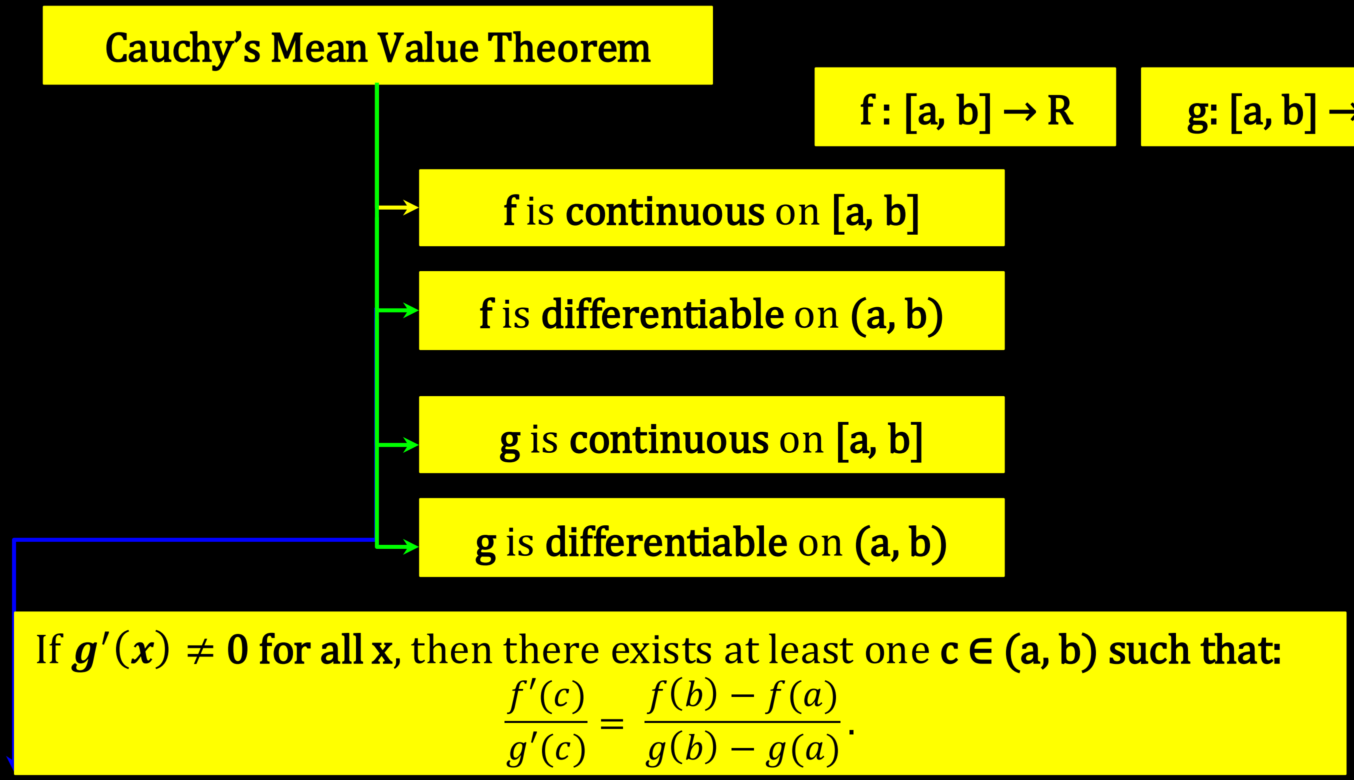
Chain Rule

Simplification

Mean Value Theorem



Extended Mean Value Theorem (Cauchy MVT)



Mean Value Theorem

Example: Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Solution: $f(-4) = (-4)^2 + 2(-4) - 8 = 0$

$f(2) = (2)^2 + 2(2) - 8 = 0$

$\because f(-4) = f(2) = 0$

$\therefore$ From Rolle's Theorem

$c \in (-4, 2)$ such that $f'(c) = 0$

$\because f(x) = x^2 + 2x - 8$

$\therefore f'(x) = 2x + 2$

$\therefore f'(c) = 2c + 2 = 0$

$\Rightarrow c = -1 \in (-4, 2)$

Polynomial Function

Continuous on $[-4, 2]$

Differentiable on $(-4, 2)$

Mean Value Theorem

Question: Verify Mean Value Theorem, if $f(x) = x^3 – 5x^2 – 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Polynomial Function
Continuous on [1, 3]
Differentiable on (1, 3)

Solution: $f(1) = (1)^3 – 5(1)^2 – 3(1) = – 7$

$f(3) = (3)^3 – 5(3)^2 – 3(3) = - 27$

$$\frac{f(b) – f(a)}{b – a} = \frac{f(3) – f(1)}{3 – 1} = \frac{-27 – (– 7)}{2} = – 10$$

From Mean Value Theorem $c \in (1, 3)$ such that

$f(x) = x^3 – 5x^2 – 3x$
$\Rightarrow f'(x) = 3x^2 – 10x – 3$

$f'(c) = – 10$

$\Rightarrow 3c^2 – 10c – 3 = – 10$

$\Rightarrow 3c^2 – 10c + 7 = 0$

Factorisation

$\Rightarrow (c – 1)(3c – 7) = 0$

$\Rightarrow c = 1, 7/3$

$c = 7/3 \in (1, 3)$ is the only point for which $f'(c) = 0$.

Mean Value Theorem

Question: If $f : [-5, 5] \rightarrow R$ is a differentiable function and $f'(x)$ does not vanish anywhere, then prove that $f(5) \neq f(-5)$.

$f'(c) \neq 0$

Continuous on $[-5, 5]$

Differentiable on $(-5, 5)$

Solution: By Mean Value Theorem

$c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

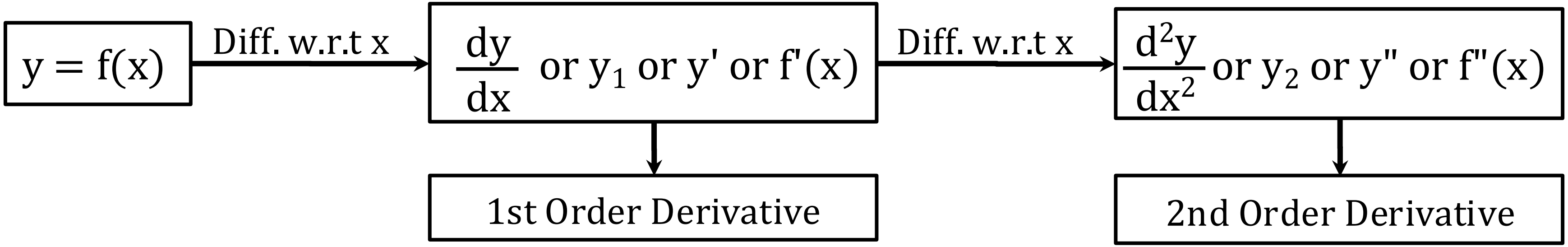
$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

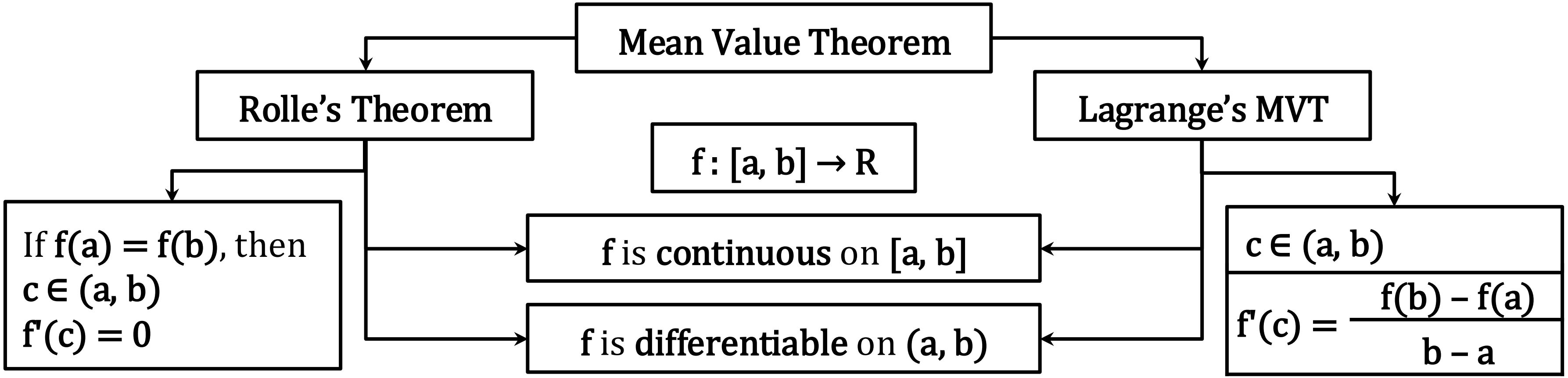
$$\therefore 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Summary


$y = f(x)$
Diff. w.r.t x
$\frac{dy}{dx}$ or y_1 or y' or $f'(x)$
1st Order Derivative
Diff. w.r.t x
$\frac{d^2y}{dx^2}$ or y_2 or y'' or $f''(x)$
2nd Order Derivative


Mean Value Theorem
Rolle's Theorem
Lagrange's MVT
$f : [a, b] \to R$
If $f(a) = f(b)$, then
$c \in (a, b)$
$f'(c) = 0$
f is continuous on $[a, b]$
f is differentiable on (a, b)
$c \in (a, b)$
$f'(c) = \frac{f(b) - f(a)}{b - a}$