Prerequisite
Integration

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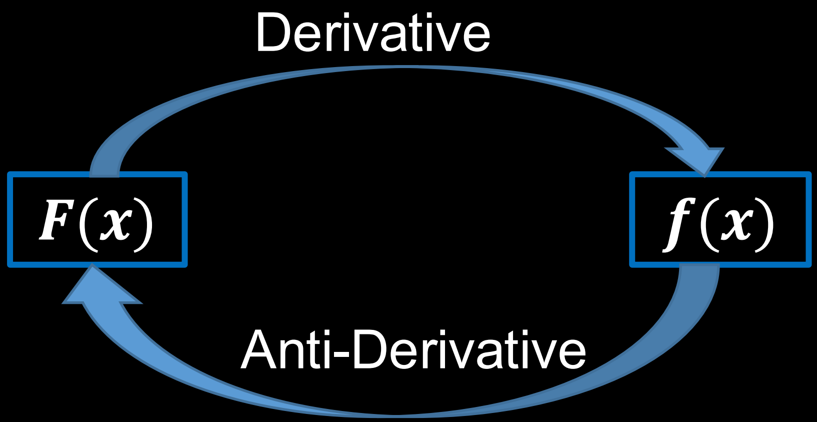
Credits: Avanti Sankalp Program



Integration

	A	B
Displacement (s)	$t^2 + 2t$	$t^3 + t^4$
Velocity (v)	$2t + 2$	$3t^2 + 4t^3$

Differentiation → $\frac{d}{dx}$

Integration → $\int$

$$\frac{d}{dx}F(x) = f(x)$$

$$\int f(x)dx = F(x)$$

Differentiation of $F(x)$ w.r.t x is equal to $f(x)$

Integration of $f(x)$ w.r.t x is equal to $F(x)$

Ex.

$$\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\, dx = \sin x$$

Fill in the Blanks

$f(x)$	$F(x)$
1	
x	
x^2	

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$n \neq -1$

$$\int \frac{1}{x}\,dx = \log x + C$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\int \cos x = \sin x$$

$$\int \sin x = -\cos x$$

$$\int \sec^2 x = \tan x$$

$$\int \sec x = ??$$

$$\int \tan x = ??$$

Constant Rule:

$$\int kf(x)dx = k\int f(x)dx$$

Addition/Subtraction Rule:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

Ex.

$$\int (2\cos x + x^2)\,dx = ?$$

$$2\int \cos x\,dx + \int x^2 dx = ?$$

$$(2\sin x + C_1) + \left(\frac{x^3}{3} + C_2\right)$$

$$2\sin x + \frac{x^3}{3} + C$$

$$\int f(x)dx = F(x) + C$$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b)$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \cos(2x+5)\, dx = \sin(2x+5) + C$$

$$\frac{d}{dx}\frac{(\sin(2x+5)+C)}{2} = \frac{2\cos(2x+5)}{2}$$

$$\int \cos(2x+5)\, dx = \frac{1}{2}\sin(2x+5) + C$$

Q. $$\int \frac{e^{2x} - 1}{e^{2x} + 1} dx$$

Q. $\int \frac{e^{2x}-1}{e^{2x}+1}dx$

Sol. $\int \frac{e^{2x}-1}{e^{2x}+1}dx$ Divide by e^x

$$\int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$$

Put, $t = e^x + e^{-x}$

$$dt = (e^x - e^{-x})dx$$

$$\int \frac{dt}{t} = \log|t| + C$$

$$= \log(e^x + e^{-x}) + C \quad ; \because e^x > 0$$

$$\log\left(e^x + \frac{1}{e^x}\right) + C$$

$$\log\left(\frac{e^{2x}+1}{e^x}\right) + C$$

$$\log(e^{2x}+1) - \log e^x + C$$

$$\log(e^{2x}+1) - x\log e + C$$

$$\log(e^{2x}+1) - x + C \quad , [\because \log e = 1]$$

$$\frac{d}{dx}(e^{2x}) = 2e^{2x} \quad , \quad \frac{d}{dx}(1) = 0$$

Q. $$\int \frac{x}{\sqrt{x+4}}\,dx$$

Q. $\int \frac{x}{\sqrt{x+4}}\,dx$

Sol. Put , $t^2 = x + 4$ or $x = t^2 - 4$

$$2t\,\frac{dt}{dx} = 1 + 0$$

$$2t\,dt = dx$$

$$\int \frac{t^2-4}{t}\,2t\,dt$$

$$2\int (t^2-4)\,dt$$

$$= 2\left[\frac{t^3}{3} - 4t\right] + C$$

$$= 2t\left[\frac{t^2}{3} - 4\right] + C$$

$$= 2\,t\left[\frac{t^2-12}{3}\right] + C$$

$$= 2\sqrt{x+4}\left[\frac{x+4-12}{3}\right] + C$$

$$= \frac{2}{3}\sqrt{x+4}\,(x-8) + C$$

Q1 $$\int \sin x \sin(\cos x)\, dx$$

Q2 $$\int (4x + 2)\sqrt{x^2 + x + 1}dx$$

Q3 $$\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$$

Q4 $$\int \frac{1}{\cos^2 x(1 - \tan x)^2} dx$$

Q5 $$\int \frac{1}{1 - \tan x} dx$$

Q1 $$\int \sin x \sin(\cos x)\, dx \quad = \cos\cos x + C$$

Q2 $$\int (4x+2)\sqrt{x^2+x+1}dx = \frac{4}{3}(x^2+x+1)^{\frac{3}{2}} + C$$

Q3 $$\int \frac{e^{\tan^{-1}x}}{1+x^2}dx \quad = e^{\tan^{-1}x} + C$$

Q4 $$\int \frac{1}{\cos^2 x(1-\tan x)^2}dx \quad = \frac{1}{1-\tan x} + C$$

Q5 $$\int \frac{1}{1-\tan x}dx \quad = \frac{x}{2} - \frac{1}{2}\log|\cos x - \sin x| + C$$

Summary

- Let $P(x) = \int \frac{f'(x)}{f(x)} dx \;\; or \;\; Q(x) = \int f(x)f'(x)\, dx$

  Steps :-

  (I) Put $t = f(x)$

  (II) $dt = f'(x)dx$

  (III) Substitute $f'(x)dx$ with dt and $f(x)$ with t

  (IV) Integrate w.r.t t.

- Some standard Integrals :-

I. $\int tan\, x\, dx = log|sec\, x| + C$

II. $\int cot\, x\, dx = log|sin\, x| + C$

III. $\int sec\, x\, dx = log|sec\, x + tan\, x| + \mathrm{C}$

IV. $\int cosec\, x\, dx = log|cosec\, x - cot\, x| + C$

Revision

$2A$ Formulae

$$\sin 2A = 2\sin A\cos A$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 2\cos^2 A - 1$$

$$= 1 - 2\sin^2 A$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$3A$ Formulae

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Product to Sum Formulae

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

Ex. $$\int \sin 6x \ \cos 3x \ dx$$

Sol.

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

Ex. $\int \sin 6x\ \cos 3x\, dx$

Sol. $\frac{1}{2}\int 2\sin 6x \cdot \cos 3x\, dx$

$\frac{1}{2}\int (\sin(6x + 3x) + \sin(6x - 3x))dx$

$\frac{1}{2}\int (\sin 9x + \sin 3x)dx$

$= \frac{1}{2}\left[\frac{-\cos 9x}{9} - \frac{\cos 3x}{3}\right] + C$

$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$

$2\cos A\sin B = \cos(A-B) - \cos(A+B)$

$2\cos A\cos B = \sin(A+B) - \sin(A-B)$

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Q1 $$\int \cos 2x \cos 4x \cos 6x \, dx$$

Q2 $$\int \frac{1-\cos x}{1+\cos x} \, dx$$

Q3 $$\int \sin^4 x \, dx$$

Q4 $$\int \frac{\cos x - \sin x}{1+\sin 2x} \, dx$$

Q5 $$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} \, dx$$

Q1 $$\int \cos 2x \cos 4x \cos 6x \, dx = \frac{1}{4}\left[\frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C$$

Q2 $$\int \frac{1 - \cos x}{1 + \cos x} dx = 2\tan\frac{x}{2} - x + C$$

Q3 $$\int \sin^4 x \, dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

Q4 $$\int \frac{\cos x - \sin x}{1 + \sin 2x} dx = -\frac{1}{\sin x + \cos x} + C$$

Q5 $$\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \tan x + C$$

Summary

- Using Trigonometric Identities convert functions like :

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad , \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4} \quad , \quad \cos^3 x = \frac{3\cos x + \cos 3x}{4}$$

- When Trigonometric ratios are given in product form, convert into sum form using these formulae

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}}$$

Put $x = a\tan\theta$; $\frac{x}{a} = \tan\theta$; $\tan\theta = \frac{\sqrt{x^2+a^2}}{a}$

$$dx = a\sec^2\theta\, d\theta$$

$$\int \frac{a\sec^2\theta\ d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$$

$$\int \frac{a\sec^2\theta\ d\theta}{a\sec\theta}$$

$$\int \sec\theta\, d\theta$$

$$\log|\sec\theta + \tan\theta| + C$$

$$\log\left|\frac{x}{a} + \frac{\sqrt{x^2 + a^2}}{a}\right| + C_1$$

$$\log\left|x + \sqrt{x^2 + a^2}\right| - \log a + C_1$$

$$\log\left|x + \sqrt{x^2 + a^2}\right| + C \quad ; \quad C = C_1 - \log a$$

Ex. $$\int \frac{dx}{1 - x^2}$$

Sol.

Ex. $$\int \frac{dx}{1 - x^2}$$

Sol. $$\int \frac{dx}{(1)^2 - x^2}$$

Put $x = a \sec \theta$

Using Standard form $$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + C$$

$$\frac{1}{2 \times 1} \log \left| \frac{1 + x}{1 - x} \right| + C$$

$$\frac{1}{2} \log \left| \frac{1 + x}{1 - x} \right| + C$$

Special forms of Integrals

$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$	$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left\|\frac{a+x}{a-x}\right\| + C$
$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left\|x + \sqrt{x^2 - a^2}\right\| + C$	$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left\|\frac{x-a}{x+a}\right\| + C$
$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left\|x + \sqrt{x^2 + a^2}\right\| + C$	$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Special forms of Integrals

$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left\|\frac{x-a}{x+a}\right\| + C$	$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left\|\frac{a+x}{a-x}\right\| + C$
$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$	$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left\|x + \sqrt{x^2 - a^2}\right\| + C$
$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$	$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left\|x + \sqrt{x^2 + a^2}\right\| + C$

Q. $$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$$

Sol.

Q. $$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$$

Sol. $$\int \frac{1}{\sqrt{(4)^2-(3+x)^2}}\,dx$$

Put, $t = 3 + x$

$$\frac{dt}{dx} = 0 + 1$$

$$dt = dx$$

$$\int \frac{dt}{\sqrt{(4)^2-t^2}} = \sin^{-1}\frac{t}{4} + C$$

$$= \sin^{-1}\frac{(3+x)}{4} + C$$

$$7-6x-x^2$$
$$= 7 - 2\times 3\times x - x^2$$
$$= 7 + 3^2 - 3^2 - 2\times 3\times x - x^2$$
$$= 16 - (3^2 + 2\times 3\times x + x^2)$$
$$= 16 - (3+x)^2$$
$$7-6x-x^2 = (4)^2 - (3+x)^2$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

Q1 $$\int \frac{1}{\sqrt{1+4x^2}}\,dx$$

Q2 $$\int \frac{3x}{1+2x^4}\,dx$$

Q3 $$\int \frac{x^2}{\sqrt{x^6+a^6}}\,dx$$

Q4 $$\int \frac{dx}{\sqrt{x^2+2x+2}}$$

Q5 $$\int \frac{5x+3}{\sqrt{x^2+4x+10}}\,dx$$

Q1 $$\int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

Q2 $$\int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}x^2)+C$$

Q3 $$\int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right|+C$$

Q4 $$\int \frac{dx}{\sqrt{x^2+2x+2}} = \log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

Q5 $$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = 5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C$$

Summary

- When $(a^2 - x^2)$ or $\frac{1}{\sqrt{a^2-x^2}}$, then put $x = a\sin\theta$ or $x = a\cos\theta$
- When $(a^2 + x^2)$ or $\frac{1}{\sqrt{a^2+x^2}}$, then put $x = a\tan\theta \text{ or } x = a\cot\theta$
- When $(x^2 - a^2)$ or $\frac{1}{\sqrt{x^2-a^2}}$, then put $x = a\sec\theta$ or $x = a\,cosec\,\theta$

Summary

- $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + C$
- $\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

Special forms of Integrals

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$
$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left\|x + \sqrt{x^2 - a^2}\right\| + C$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log\left\|x + \sqrt{x^2 + a^2}\right\| + C$

Principle

$\int \frac{1}{x} \log x \, dx$ Substitution

$\int \sin^3 x \, dx$ Trigonometrical Identities

$\int \frac{1}{(x+1)(x-2)} dx$ Partial Fractions

$\int \sin^{-1} x \ln x \, dx = ?$

$\int x \, e^x dx = ?$

Anti-derivative of product rule

$$\int U.V\,dx = U\int V\,dx - \int\left(\frac{dU}{dx}\right)\left(\int V\,dx\right)dx$$

ILATE rule

$\int U.V \ dx$

$$\int x \cos x \ dx$$

I	→	Inverse Function	$\sin^{-1} x$
L	→	Logarithm Function	$\ln x$
A	→	Algebraic Function	x^3
T	→	Trigonometric Function	$\sin x$
E	→	Exponential Function	e^x

$$\int x \sin^{-1} x \, dx$$

$V \quad U$

I L A T E

Q. Find $\int x \cos x \, dx$

Sol.

Q. Find $\int x \cos x \, dx$

Sol.

$$\int \underset{U}{x} \, \underset{V}{\cos x} \, dx$$

I L A T E

$$x \int \cos x \, dx - \int \int cosx \;\; dx \left(\frac{dx}{dx}\right) dx$$

$$x \sin x - \int \sin x \, dx$$

$$x \sin x + \cos x + C$$

Q. Find $\int \log x \, dx$

Sol.

Q. Find $\int \log x \, dx$

Sol.

$$\int \log x \, . \, 1 . \, dx$$

$$\log x \int 1 . \, dx - \int \frac{d(\log x)}{dx} \int 1 . \, dx \, dx$$

$$x \log x - \int \frac{1}{x} x \, dx$$

$$x \log x - x + C$$

Q. Evaluate the **definite integrals,** if

1. $\int_0^1 \frac{dx}{1+x^2}$

2. $\int_0^1 x e^{x^2}\, dx$

Q. Evaluate the **definite integrals,** if

1. $\int_0^1 \frac{dx}{1+x^2}$

2. $\int_0^1 xe^{x^2}\,dx$

Sol.

$$= \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}$$

$$= \int_0^1 e^{x^2} x\,dx$$

$$e^{x^2} = t$$

$$e^{x^2} 2x\,dx = dt$$

$$e^{x^2} x\,dx = \frac{dt}{2}$$

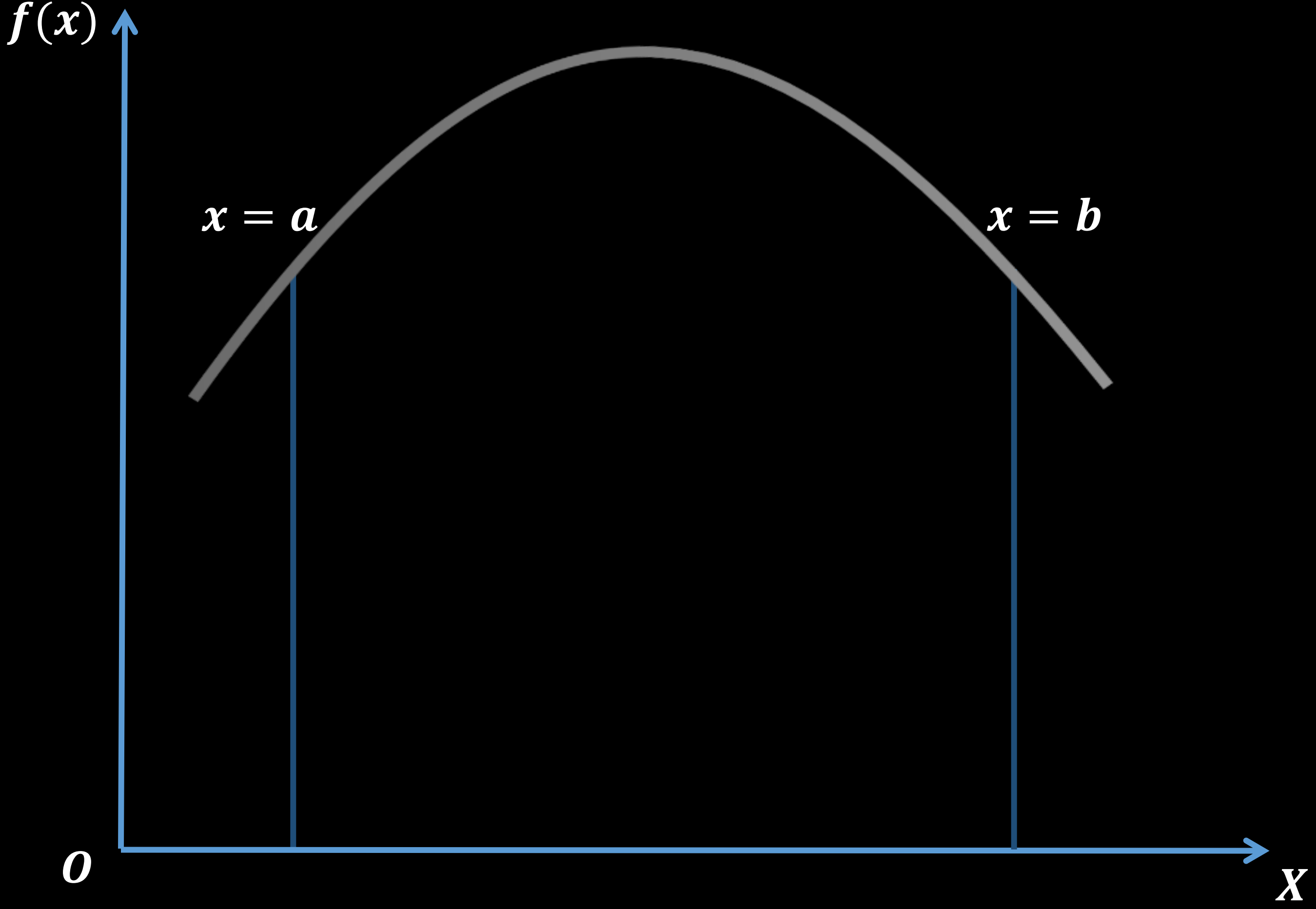
$x = 0 \quad t = 1$

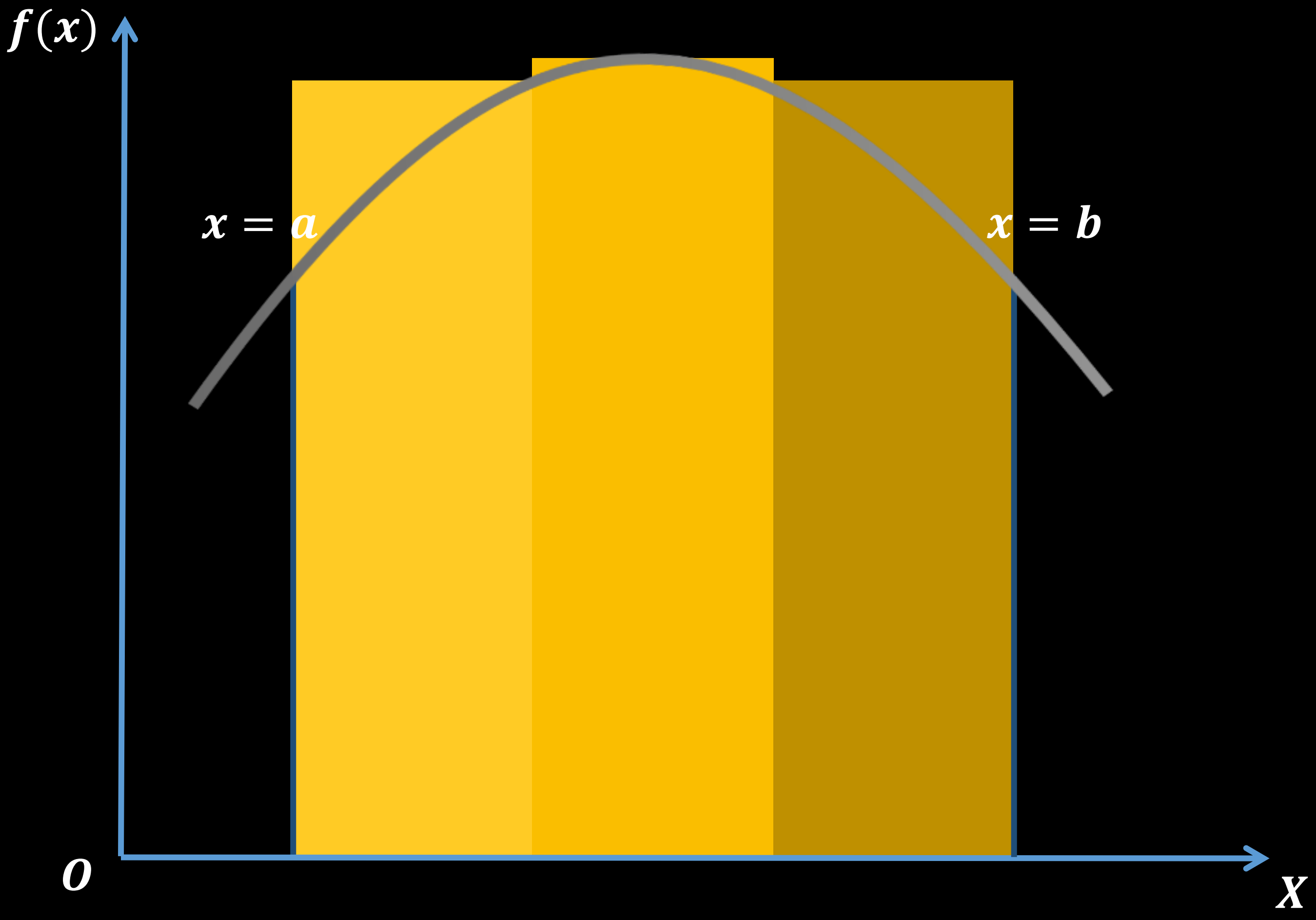
$x = 1 \quad t = e$

$$= \int_1^e \frac{dt}{2}$$

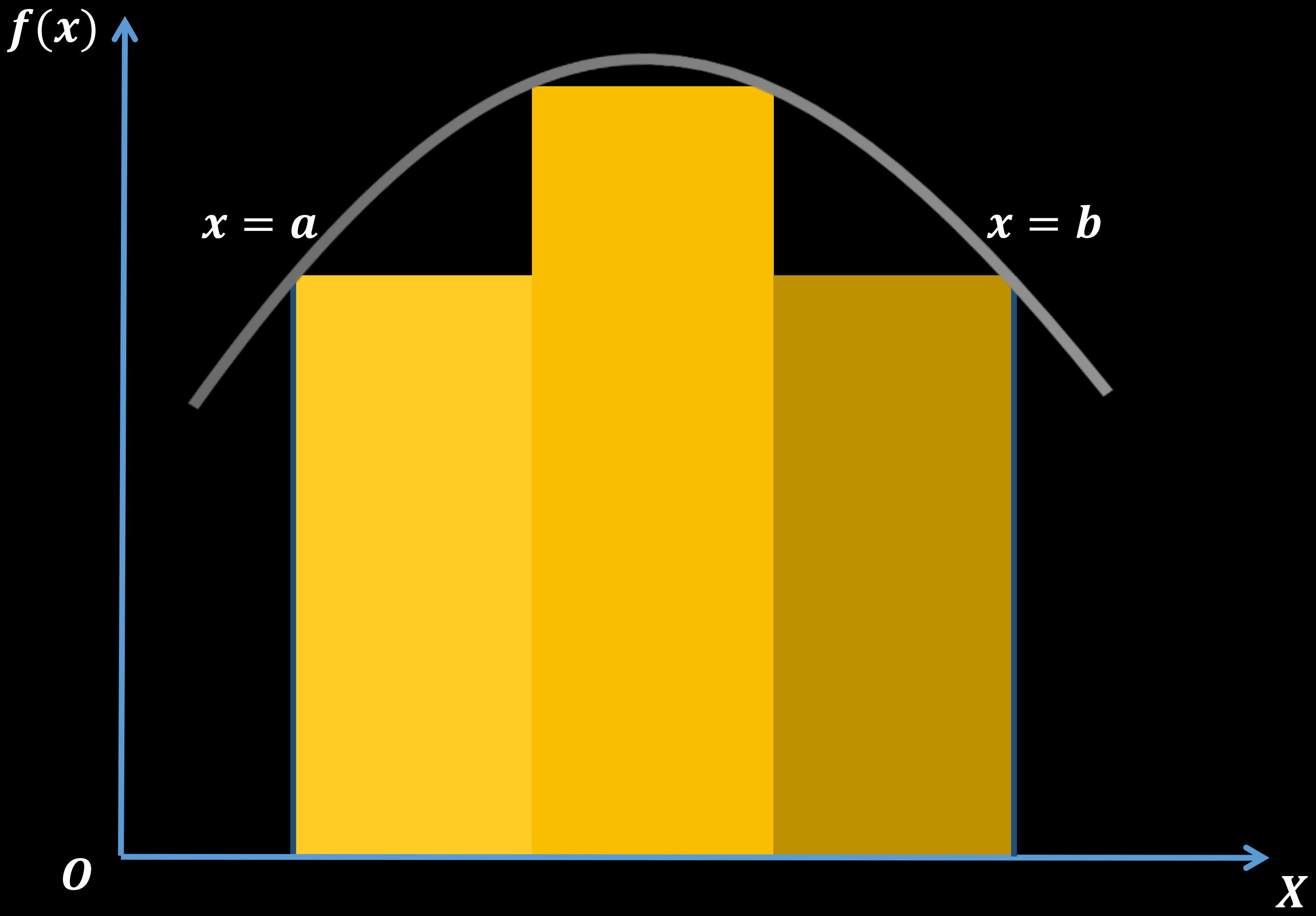
$$= \left[\frac{t}{2}\right]_1^e$$

$$= \frac{1}{2}[e-1]$$

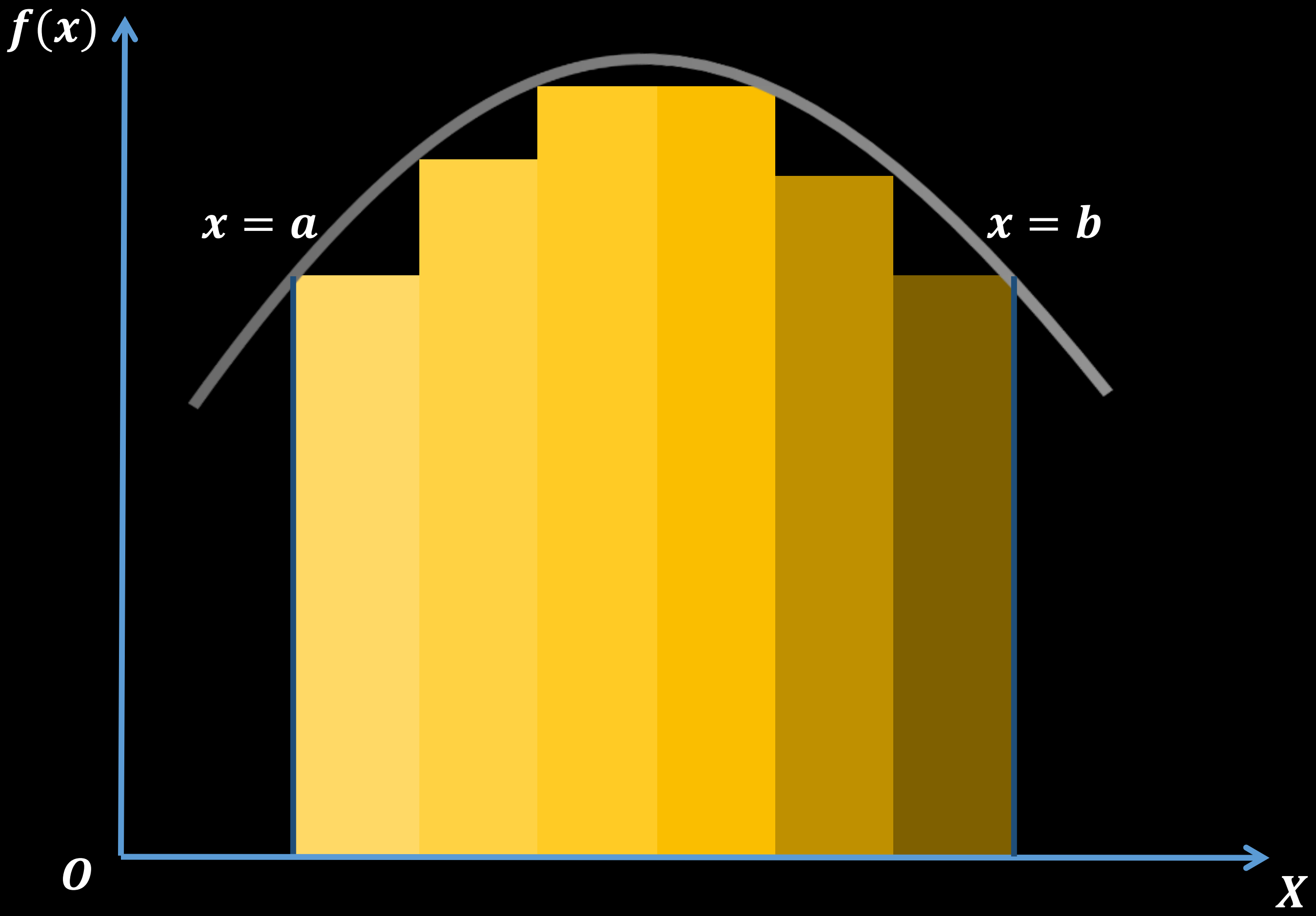

$f(x)$
$x = a$
$x = b$
O
X



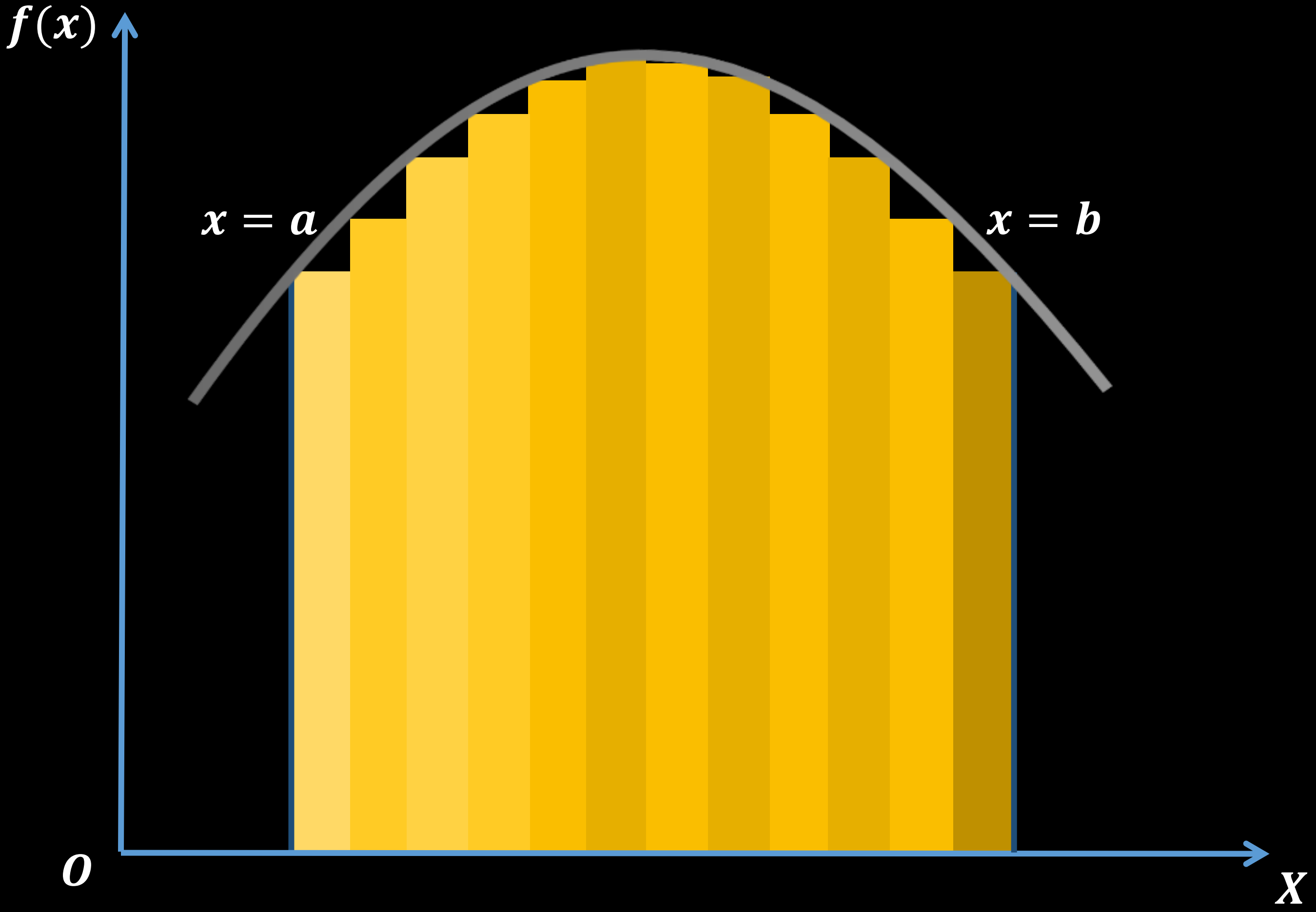
$f(x)$
$x = a$
$x = b$
O
X

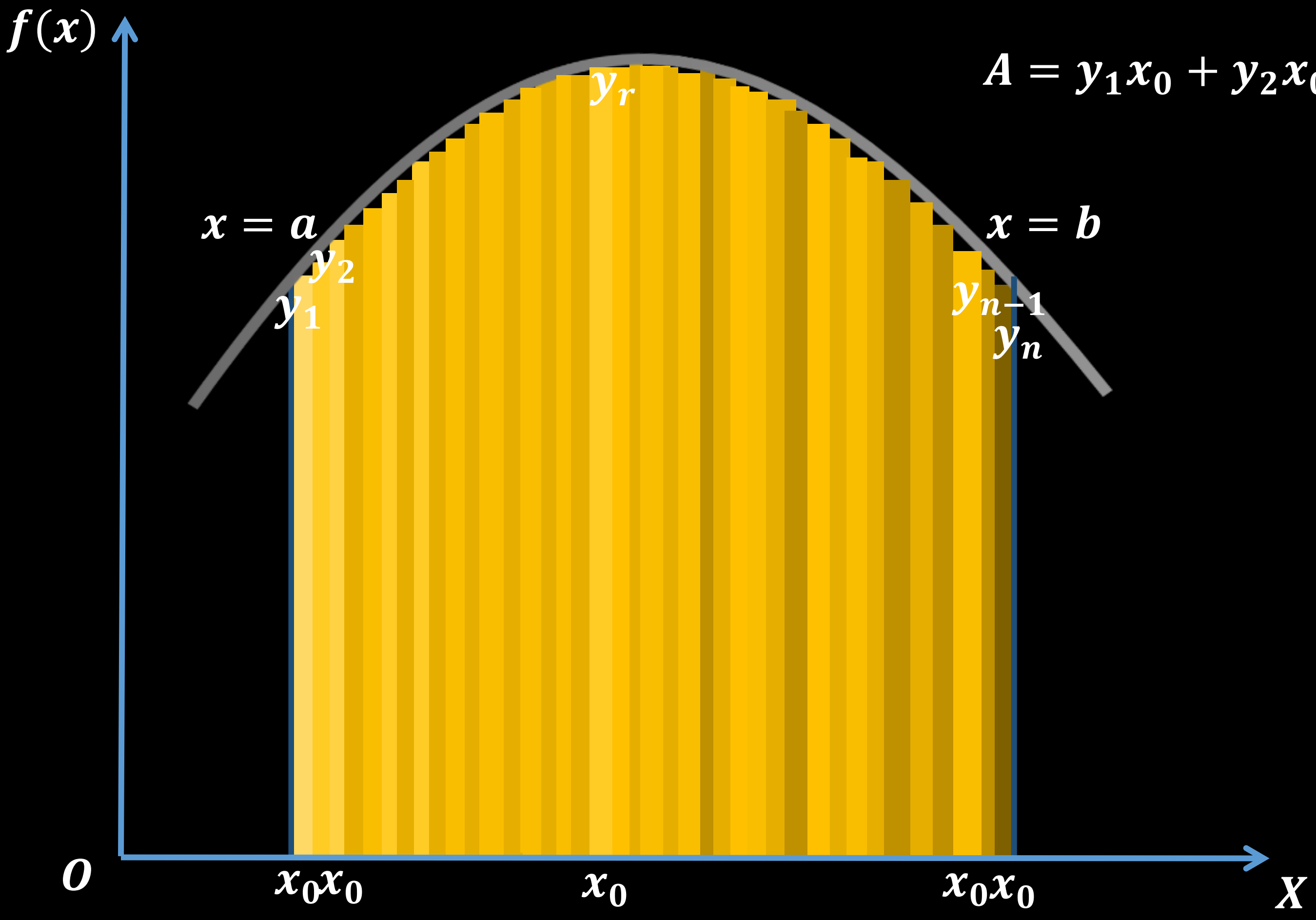


f(x)
x = a
x = b
O
X

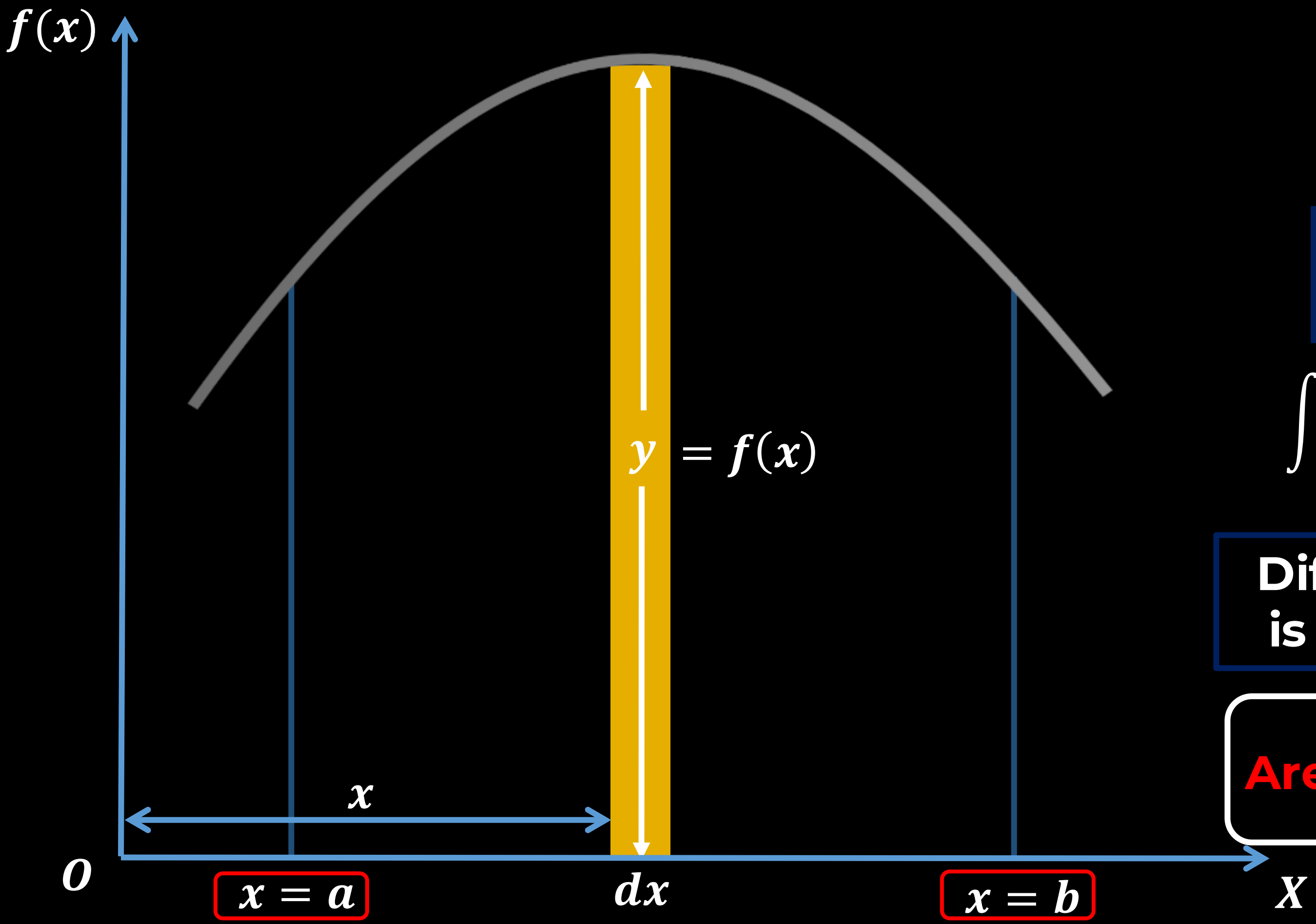


f(x)
x = a
x = b
O
X


$f(x)$
$x = a$
$x = b$
O
X



$$A = y_1 x_0 + y_2 x_0 + \cdots y_r x_0 + \cdots + y_n x_0$$



$$dA = y\,dx$$

$$dA = f(x)dx$$

$$\boxed{\frac{dA}{dx} = f(x)}$$

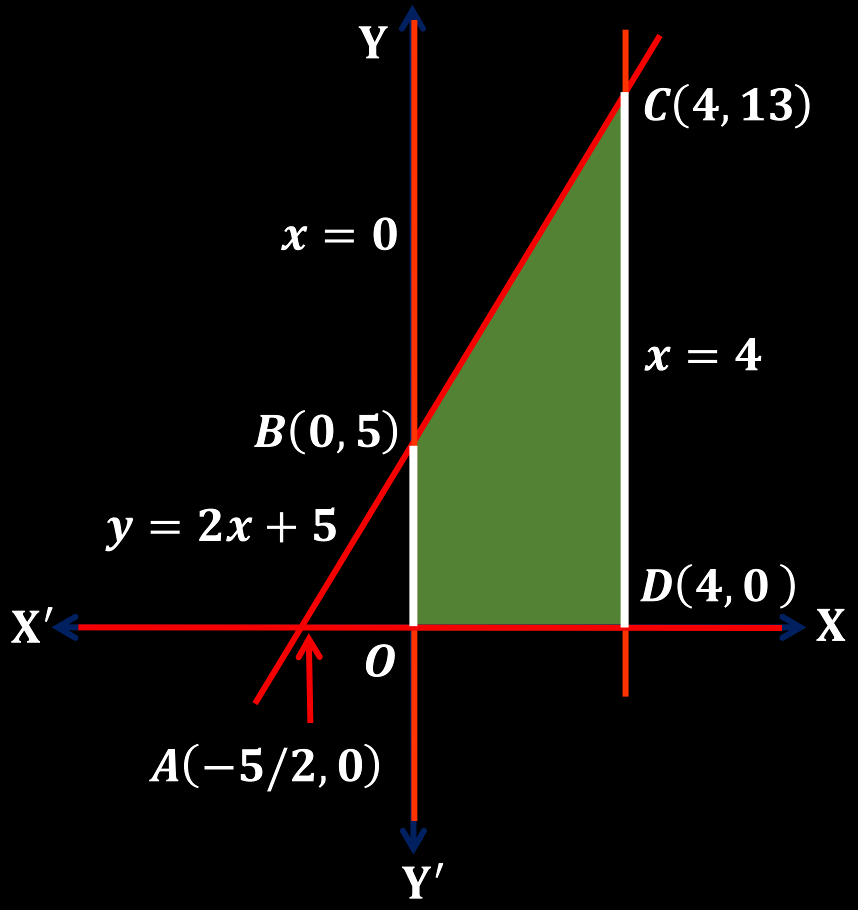
$$\int f(x)\,dx = A$$

Diff. of A w.r.t. x is equal to $f(x)$

Area $= \int_a^b f(x)\,dx$

Q. Find the area of the region bounded by $y = 2x + 5$ and x-axis for $x = 0$ to $x = 4$ in first quadrant.

Sol.



Q. Find the area of the region bounded by $y = 2x + 5$ and x-axis for $x = 0$ to $x = 4$ in first quadrant.

Sol.

$$Area = \int_a^b f(x)\,dx$$

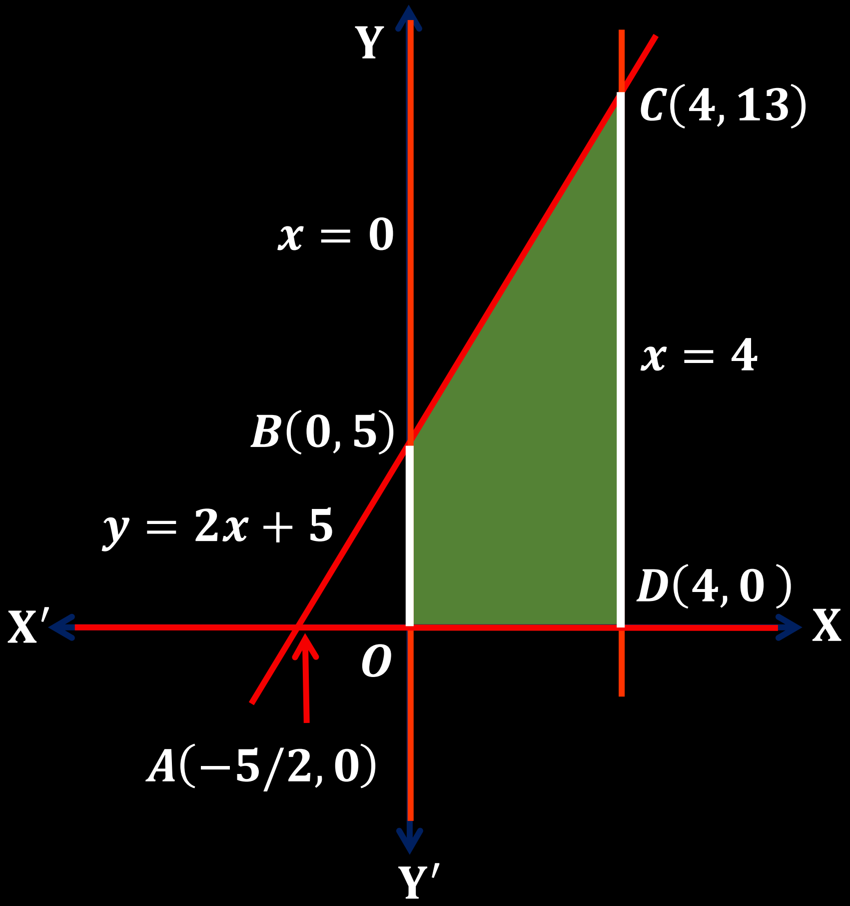
$f(x) = y = 2x + 5 \qquad a = 0 \qquad b = 4$

$$Area\ of\ BCDOB = \int_0^4 (2x + 5)\,dx$$

$$= \left[x^2 + 5x\right]_0^4$$

$$= \left[(4)^2 + 5(4)\right] - \left[(0)^2 + 5(0)\right]$$

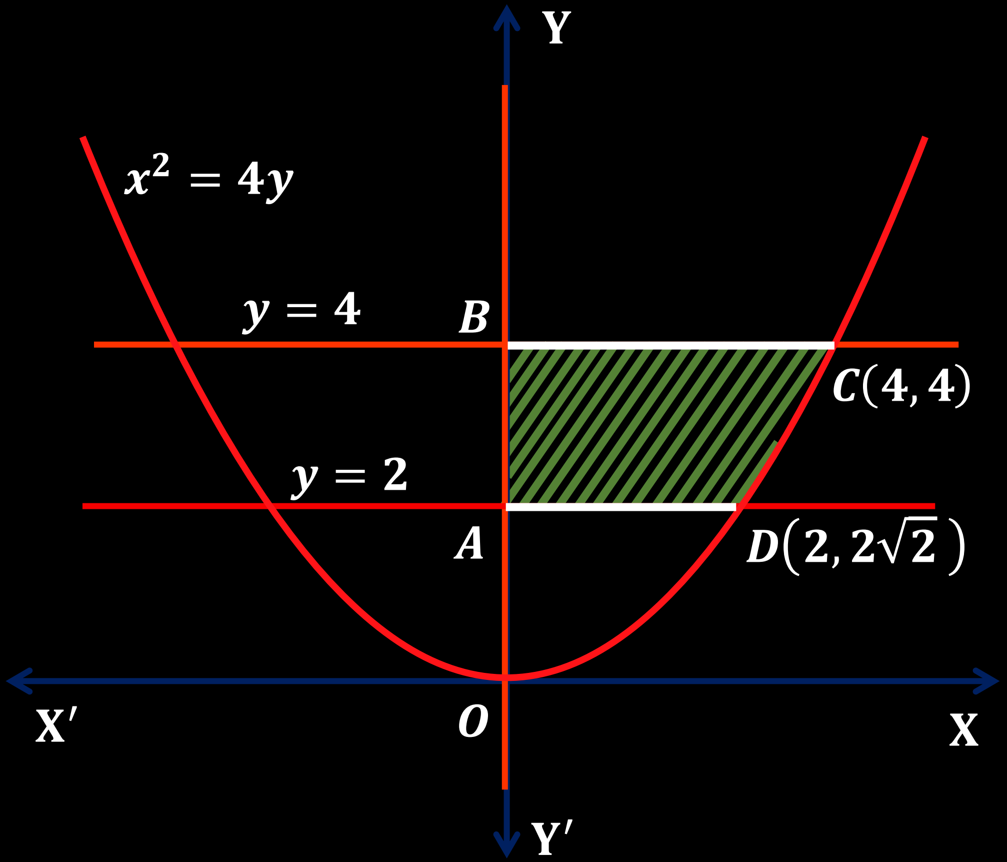
$= 36$ units



Q. **Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.**

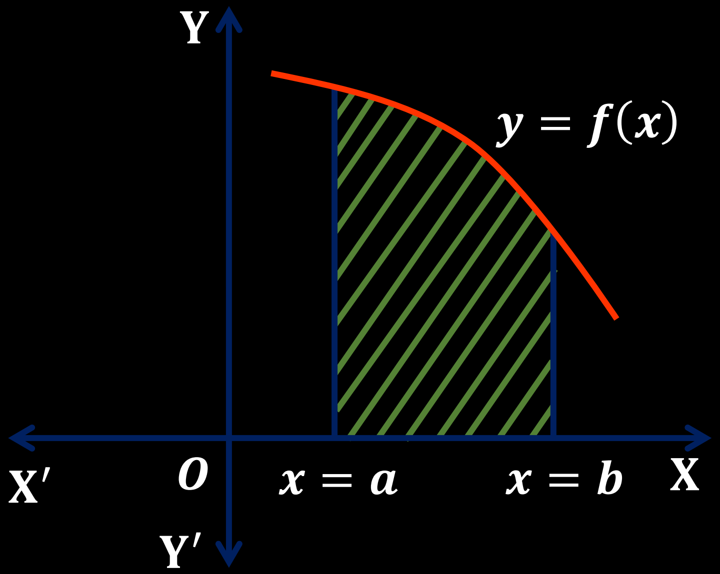
Sol.

Sometimes, changing the perspective matters.



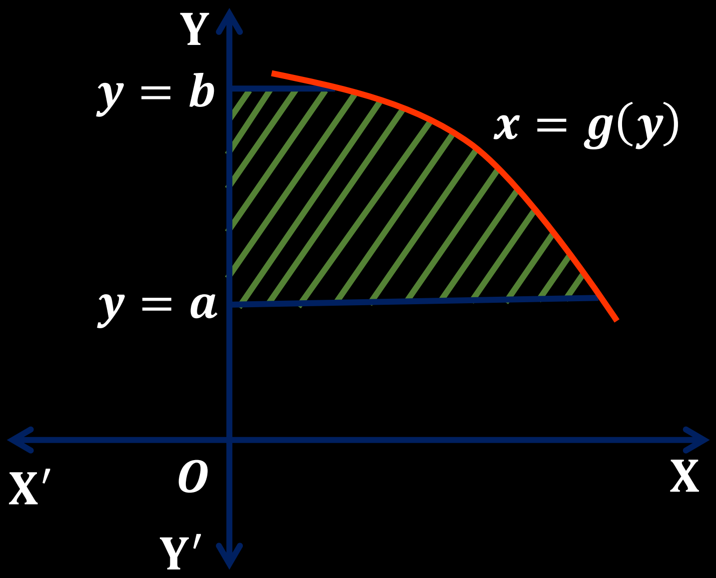
$$A = \int_a^b f(x)\,dx$$

The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$.



$$A = \int_a^b g(y)\,dy$$

The area of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = a$ and $y = b (b > a)$.



Q. Find the area of the region bounded by $x^2 = 4y$, $y = 2$, $y = 4$ and the y-axis in the first quadrant.

Sol.

$$A = \int_a^b g(y)\,dy$$

$x^2 = 4y \Rightarrow x = \pm 2\sqrt{y}$

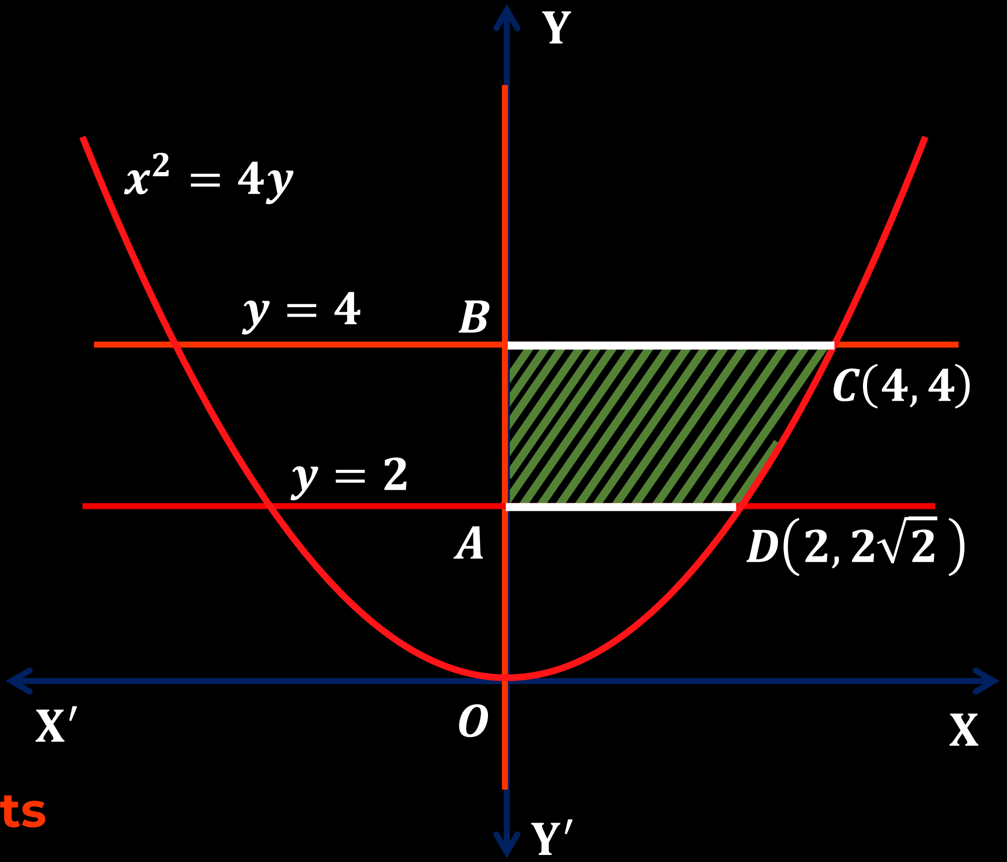
$g(y) = 2\sqrt{y} \quad a = 2 \quad b = 4$

$$A = \int_2^4 2\sqrt{y}\,dy = \left[2 \times \frac{y^{3/2}}{3/2}\right]_2^4$$

$$= \frac{4}{3}\left[y^{3/2}\right]_2^4$$

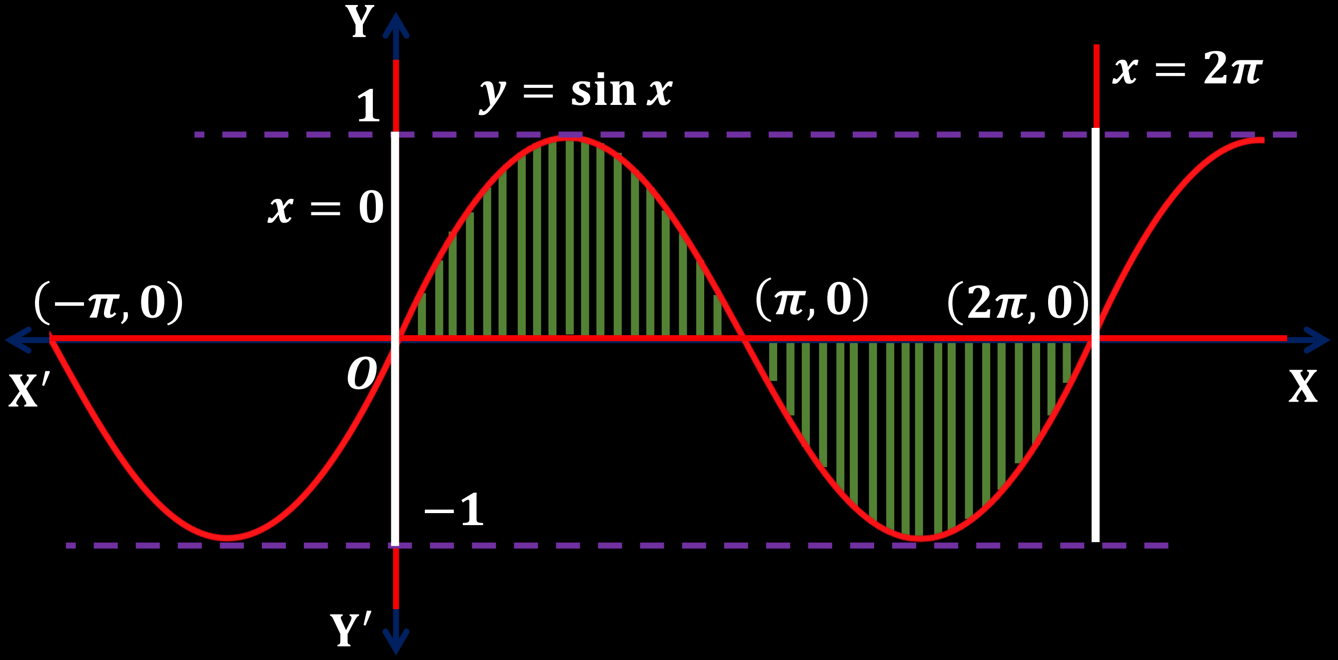
$$= \frac{4}{3}\left[\left(4^{3/2}\right) - \left(2^{3/2}\right)\right]$$

$$= \frac{4}{3}\left[8 - 2\sqrt{2}\right] = \frac{32 - 8\sqrt{2}}{3}$$ **Units**



Q. Find the area of the region bounded by $y = \sin x$ and x-axis for $x = 0$ to $x = 2\pi$.

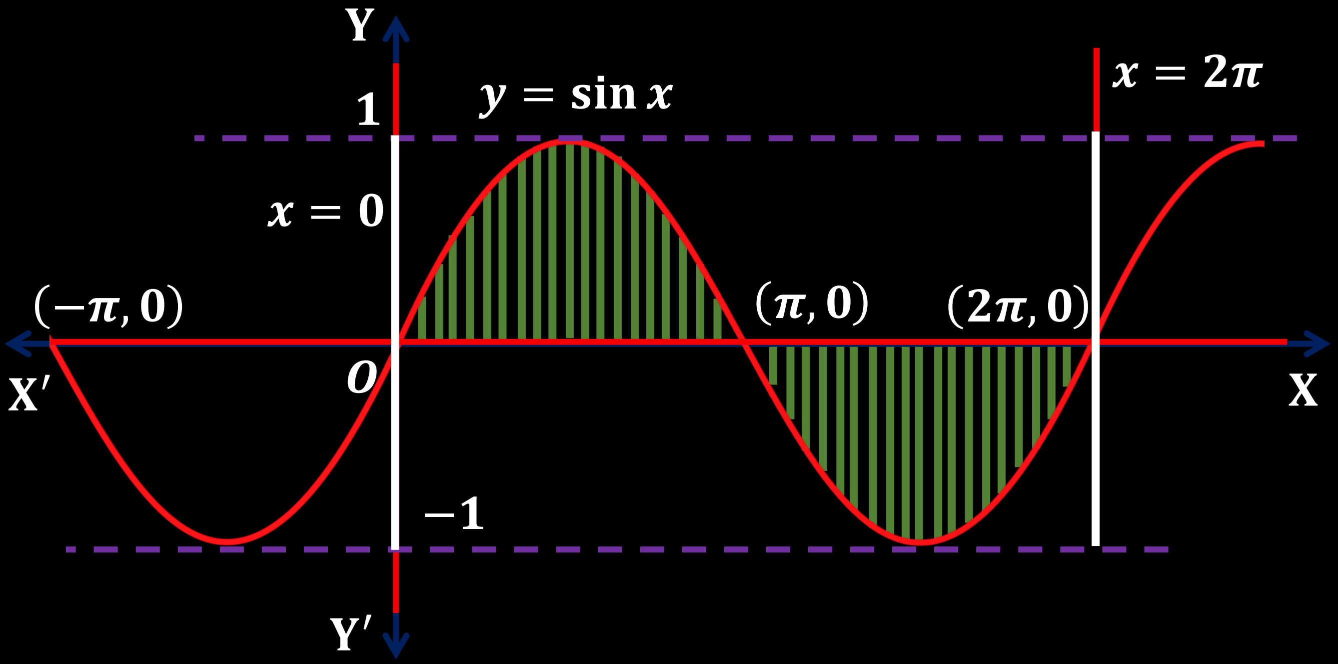
Sol.



Q. Find the area of the region bounded by $y = \sin x$ and x-axis for $x = 0$ to $x = 2\pi$.

Sol.

$$A = \int_a^b f(x)\,dx$$



$$f(x) = y = \sin x$$

$$a = 0 \qquad b = 2\pi$$

$$A = \int_0^{2\pi} \sin x\,dx$$

$$= [-\cos x]_0^{2\pi}$$

$$= (-\cos 2\pi) - (-\cos 0)$$

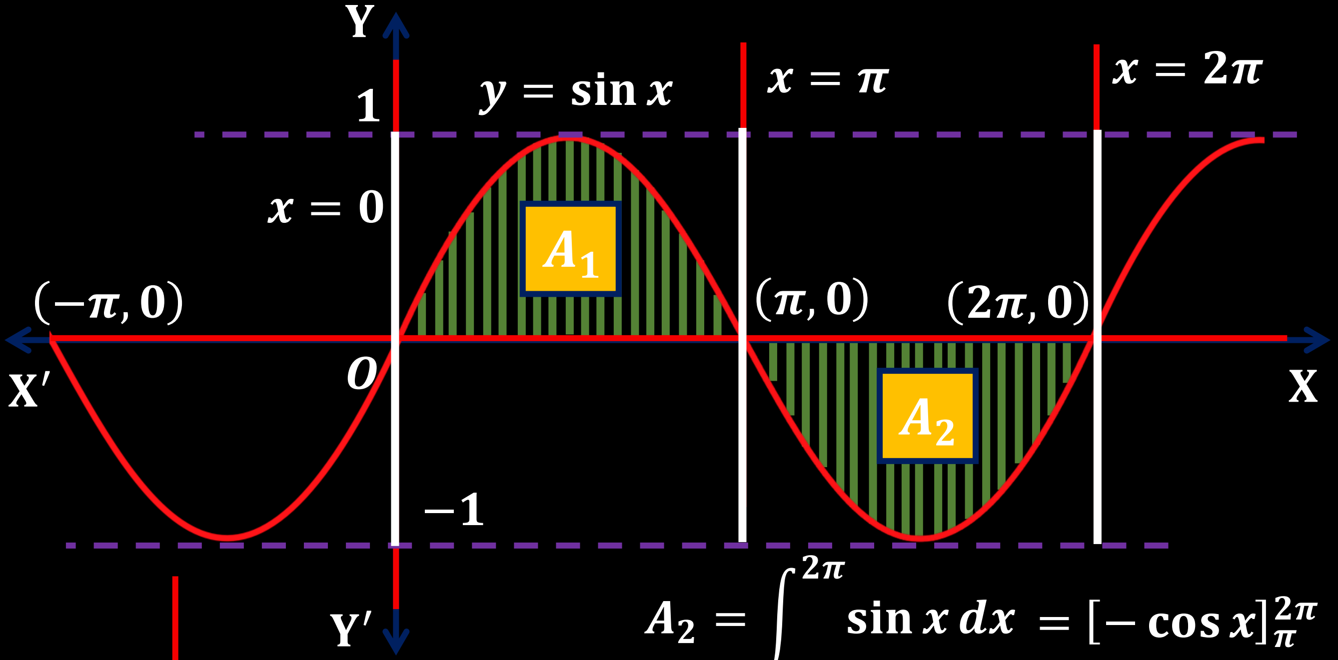
$$= (-1) - (-1)$$

$$= 0$$

Q. Find the area of the region bounded by $y = \sin x$ and x-axis for $x = 0$ to $x = 2\pi$.

Sol.

$$A = \int_a^b f(x)\,dx$$



$f(x) = y = \sin x$

$a = 0 \quad b = \pi$

$$A_1 = \int_0^\pi \sin x\,dx$$

$= [-\cos x]_0^\pi$

$= (-\cos \pi) - (-\cos 0)$

$= \big(-(-1)\big) - (-1)$

$= 2$ **Units**

$f(x) = y = \sin x$

$a = \pi \quad b = 2\pi$

$$A_2 = \int_\pi^{2\pi} \sin x\,dx = [-\cos x]_\pi^{2\pi}$$

$= (-\cos 2\pi) - (-\cos \pi)$

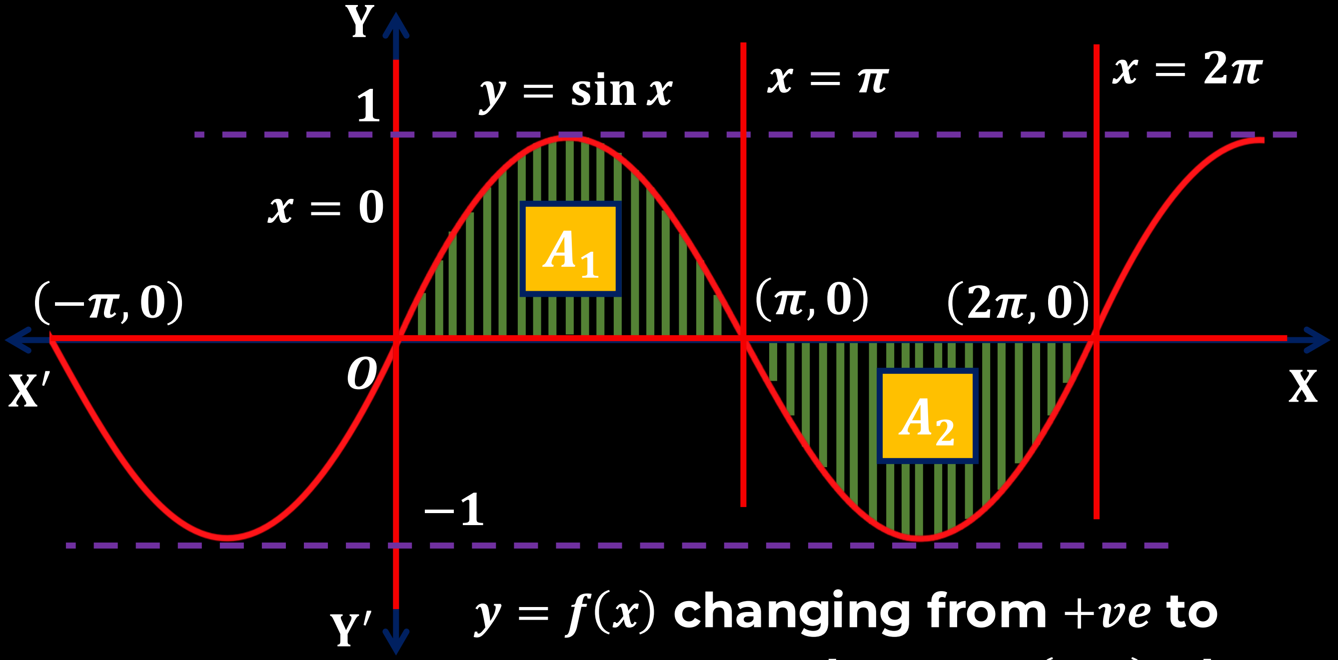
$= \big(-(1)\big) - \big(-(-1)\big)$

$= -2$ **Units**

Q. Find the area of the region bounded by $y = \sin x$ and x-axis for $x = 0$ to $x = 2\pi$.

Sol.

$$A = \int_a^b f(x)\,dx$$



$A_1 = 2$ Units

$A_2 = -2$ Units

$|A_2| = 2$ Units

$A = A_1 + |A_2|$

$= 4$ Units

$y = f(x)$ changing from $+ve$ to $-ve$ at $x = c$, where $c \in (a, b)$, then

$$A = \left|\int_a^b f(x)\,dx\right|$$

$$A = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|$$

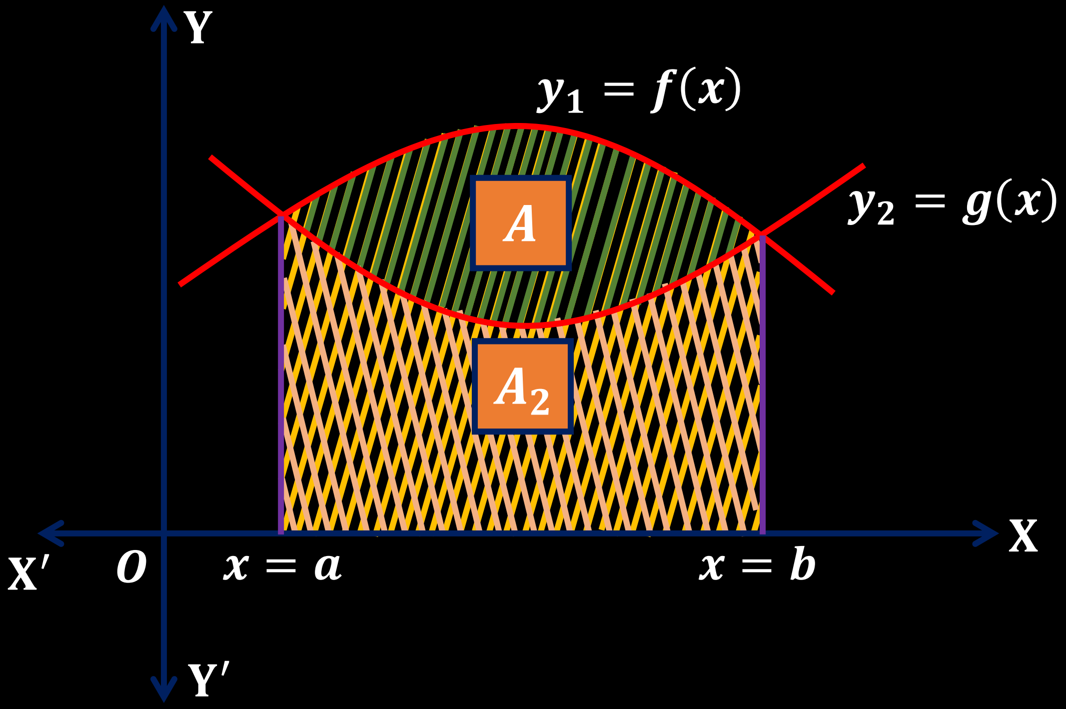
Summary

The area of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ $(b > a)$ is given by the formula:

$$Area = \left| \int_a^b f(x)\,dx \right|$$

The area of the region bounded by the curve $x = g(y)$, y-axis and the lines $y = c$ and $y = d (d > c)$ is given by the formula

$$Area = \left| \int_c^d g(y)\,dy \right|$$



$$A_1 = \left|\int_a^b f(x)\,dx\right|$$

$$A_2 = \left|\int_a^b g(x)\,dx\right|$$

$$A = A_1 - A_2$$

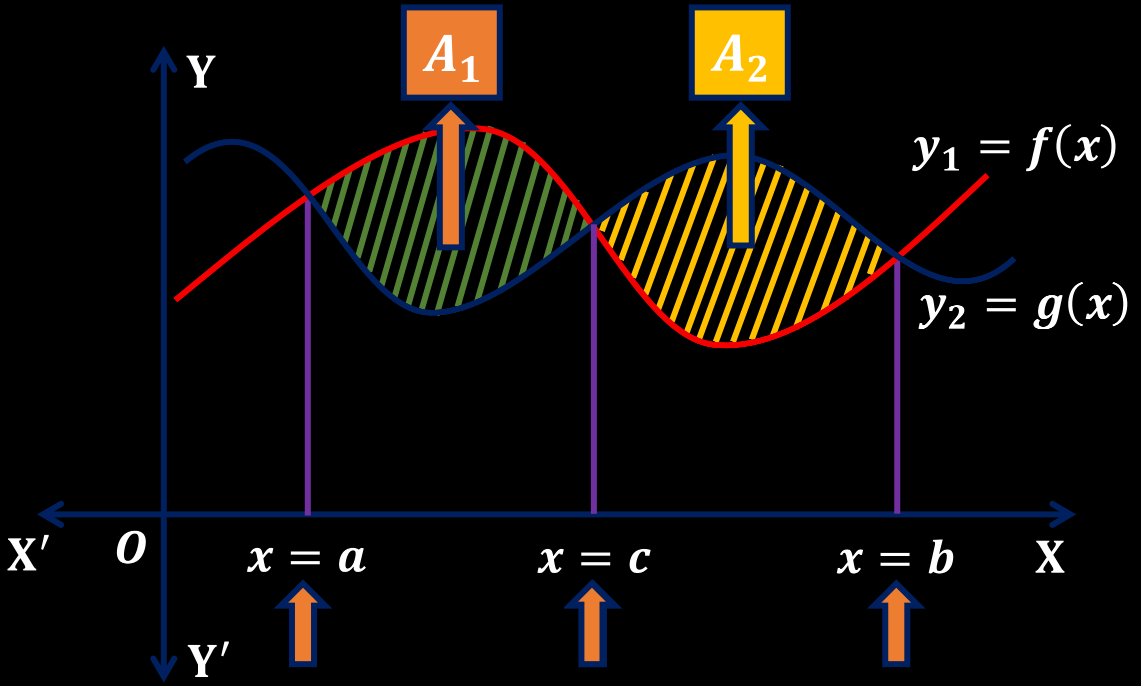
$$A = \left|\int_a^b f(x)\,dx\right| - \left|\int_a^b g(x)\,dx\right|$$

$$A = \left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right|$$

$$A = \left|\int_a^b (Upper\ Curve - Lower\ Curve)\,dx\right|$$

$$A = \left|\int_a^b \big(f(x) - g(x)\big)\,dx\right|$$

$$A = \left| \int_a^b (Upper\ Curve - Lower\ Curve)\, dx \right|$$



For A_1

$Upper\ Curve = f(x)$

$Lower\ Curve = g(x)$

$$A_1 = \left| \int_a^c (f(x) - g(x))\, dx \right|$$

For A_2

$Upper\ Curve = g(x)$

$Lower\ Curve = f(x)$

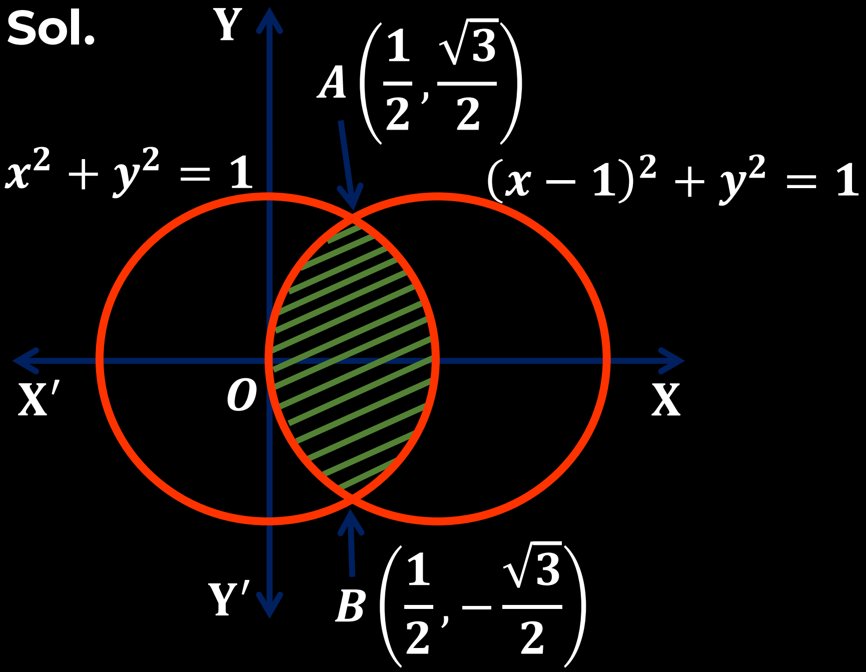
$$A_2 = \left| \int_c^b (g(x) - f(x))\, dx \right|$$

$A = A_1 + A_2$

$$A = \left| \int_a^c (f(x) - g(x))\, dx \right| + \left| \int_c^b (g(x) - f(x))\, dx \right|$$

Q. Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.

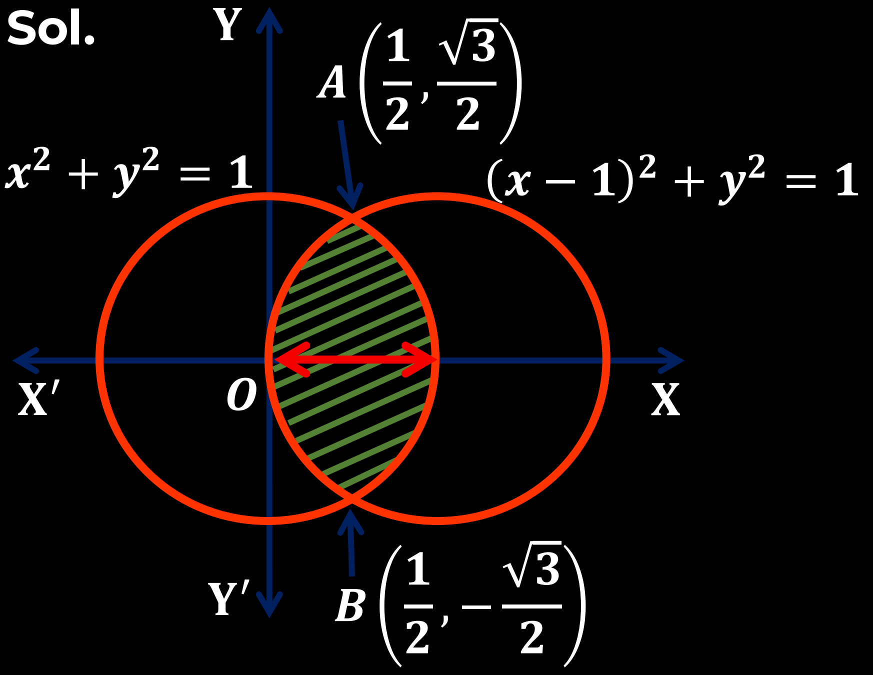
Sol.



$$A=\left|\int_a^b \left(f(y)-g(y)\right)dy\right|$$

Q. Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.

Sol.



$$A=\left|\int_a^b \left(f(y)-g(y)\right)dy\right|$$

$f(y)=\sqrt{1-y^2}$ $\quad g(y)=1-\sqrt{1-y^2}$

$$x^2+y^2=1$$

$$\Rightarrow x^2=1-y^2$$

$$\Rightarrow x=\pm\sqrt{1-y^2}$$

$$(x-1)^2+y^2=1$$

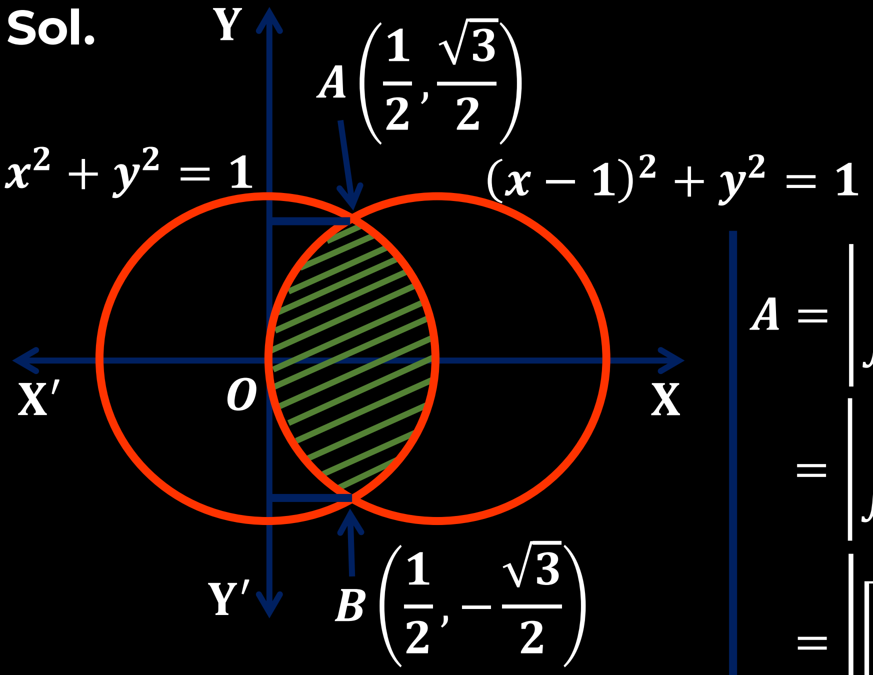
$$\Rightarrow (x-1)^2=1-y^2$$

$$\Rightarrow x-1=\pm\sqrt{1-y^2}$$

$$\Rightarrow x=1\pm\sqrt{1-y^2}$$

Q. Find the area bounded by curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.

Sol.



$$A=\left|\int_a^b (f(y)-g(y))\,dy\right|$$

$$f(y)=\sqrt{1-y^2}\quad g(y)=1-\sqrt{1-y^2}$$

$$a=-\frac{\sqrt{3}}{2} \text{ and } b=\frac{\sqrt{3}}{2}$$

$$A=\left|\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left(\sqrt{1-y^2}-\left(1-\sqrt{1-y^2}\right)\right)dx\right|$$

$$=\left|\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\left(2\sqrt{1-y^2}-1\right)dx\right|$$

$$=\left|\left[2\left(\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\sin^{-1}y\right)-y\right]_{-\sqrt{3}/2}^{\sqrt{3}/2}\right|$$

$$A=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\text{ Units}$$

Summary

The area of the region bounded by the **curve** $y = f(x), y = g(x)$ and the **lines** $x = a$ and $x = b\ (b > a)$ is given by the formula:

$$Area = \left| \int_a^b \left(f(x) - g(x) \right) dx \right|$$

where $f(x) \geq g(x)$ in $[a, b]$

If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b], a < c < b$, then

$$Area = \left| \int_a^c \left(f(x) - g(x) \right) dx \right| + \left| \int_c^b \left(g(x) - f(x) \right) dx \right|$$