Triple Integral over a Tetrahedral Region

A Complete Cartesian Solution

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1. Problem Statement

Evaluate the triple integral:

$$\iiint\_V (x + y + z)\ dV$$

where V is the region bounded by the four planes:

- $x = 0$ (yz-plane)
- $y = 0$ (xz-plane)
- $z = 0$ (xy-plane)
- $2x + 3y + 4z = 12$ (slanted plane)

2. Geometric Analysis — The Tetrahedron

The four planes form a tetrahedron in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). The vertices are found by intersecting the coordinate planes with $2x + 3y + 4z = 12$:

$O = (0, 0, 0)$ (origin)

$A = (6, 0, 0)$ (set $y = 0$, $z = 0$: $2x = 12 \Rightarrow x = 6$)

$B = (0, 4, 0)$ (set $x = 0$, $z = 0$: $3y = 12 \Rightarrow y = 4$)

$C = (0, 0, 3)$ (set $x = 0$, $y = 0$: $4z = 12 \Rightarrow z = 3$)

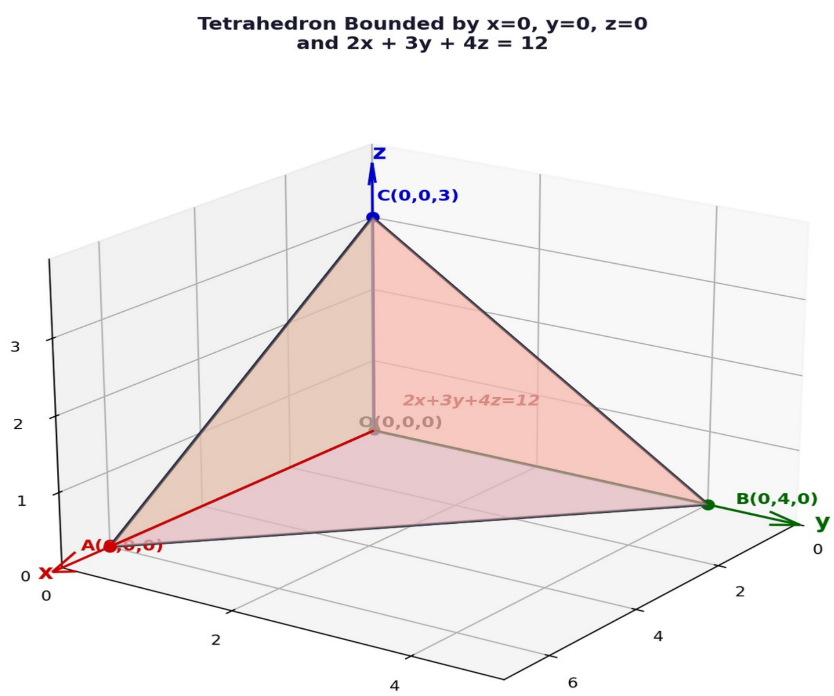

Figure 1: Tetrahedron V bounded by x = 0, y = 0, z = 0, and 2x + 3y + 4z = 12

3. Establishing Limits of Integration

We integrate in the order dz dy dx. For each level of integration, we solve for the upper limit from the plane equation.

Step 1 — Limits for z

For fixed x and y, z ranges from 0 up to the slanted plane. Solving 2x + 3y + 4z = 12 for z:

$$4z = 12 - 2x - 3y$$

$$z = (12 - 2x - 3y) / 4 = 3 - x/2 - 3y/4$$

Therefore:

$$\mathbf{0 \ \leq \ z \ \leq \ 3 - x/2 - 3y/4}$$

Step 2 — Limits for y

For fixed x, y ranges from 0 to the value where the upper z-limit becomes zero (the slanted plane touches z = 0):

$$3 - x/2 - 3y/4 = 0 \quad \Rightarrow \quad y = (4/3)(3 - x/2) = 4 - 2x/3$$

Therefore:

$$\mathbf{0 \ \leq \ y \ \leq \ 4 - 2x/3}$$

Step 3 — Limits for x

x ranges from 0 to the x-intercept of the plane (where y = 0 and z = 0):

$$2x = 12 \quad \Rightarrow \quad x = 6$$

$$\mathbf{0 \ \leq \ x \ \leq \ 6}$$

Full Integral Setup

$$\mathbf{\int_0^6 \ \int_0\hat{}(4{-}2x/3) \ \int_0\hat{}(3{-}x/2{-}3y/4) \ (x + y + z) \ dz \ dy \ dx}$$

4. Evaluation of the Triple Integral

Step 4 — Innermost Integral: Integrate with Respect to z

Let U = 3 − x/2 − 3y/4 for brevity. We compute:

$$\int_0\hat{}U \ (x + y + z) \ dz \ = \ [(x + y)z + z^2/2]_0\hat{}U$$

$$= \ (x + y)U + U^2/2$$

$$= \ U(x + y) + U^2/2$$

Expanding with U = 3 − x/2 − 3y/4:

$$= (3 - x/2 - 3y/4)(x + y) + (3 - x/2 - 3y/4)^2/2$$

Expanding the first product:

$$(3 - x/2 - 3y/4)(x + y) = 3x + 3y - x^2/2 - xy/2 - 3xy/4 - 3y^2/4$$
$$= 3x + 3y - x^2/2 - 5xy/4 - 3y^2/4$$

Expanding the second term $U^2/2$:

$$U^2 = (3 - x/2 - 3y/4)^2 = 9 - 3x - 9y/2 + x^2/4 + 3xy/4 + 9y^2/16$$
$$U^2/2 = 9/2 - 3x/2 - 9y/4 + x^2/8 + 3xy/8 + 9y^2/32$$

Adding both parts (inner integral result I_1):

$$I_1 = 3x + 3y - x^2/2 - 5xy/4 - 3y^2/4 + 9/2 - 3x/2 - 9y/4 + x^2/8 + 3xy/8 + 9y^2/32$$
$$I_1 = 9/2 + 3x/2 + 3y/4 - 3x^2/8 - 7xy/8 - 15y^2/32$$

Step 5 — Middle Integral: Integrate with Respect to y

Let V_x = 4 − 2x/3 (upper limit for y). We integrate I_1 from y = 0 to y = V_x:

$$\int_0\hat{}(V\_x)\ I_1\ dy\ =\ \int_0\hat{}(V\_x)\ [9/2 + 3x/2 + 3y/4 - 3x^2/8 - 7xy/8 - 15y^2/32]\ dy$$

Integrating term by term:

$$= [9y/2 + 3xy/2 + 3y^2/8 - 3x^2y/8 - 7xy^2/16 - 5y^3/32]_0\hat{}(V\_x)$$

Substituting y = 4 − 2x/3 and simplifying (collecting powers of x):

After careful substitution and algebraic expansion of each term:

$$= (9/2)(4–2x/3) + (3x/2)(4–2x/3) + (3/8)(4–2x/3)^2$$
$$- (3x^2/8)(4–2x/3) - (7x/16)(4–2x/3)^2 - (5/32)(4–2x/3)^3$$

Let w = 4 − 2x/3. Expanding each term systematically:

$$w = 4 - 2x/3$$
$$w^2 = 16 - 16x/3 + 4x^2/9$$
$$w^3 = 64 - 96x/3 + 48x^2/9 - 8x^3/27 = 64 - 32x + 16x^2/3 - 8x^3/27$$

Computing each term:

$$(9/2)w = 18 - 3x$$
$$(3x/2)w = 6x - x^2$$
$$(3/8)w^2 = 6 - 2x + x^2/6$$
$$(3x^2/8)w = 3x^2/2 - x^3/4$$
$$(7x/16)w^2 = 7x - 7x^2/3 + 7x^3/36$$
$$(5/32)w^3 = 10 - 5x + 5x^2/6 - 5x^3/108$$

Collecting all terms (I_2 = result after y-integration):

$$I_2 = (18 + 6 - 10) + (-3 + 6 - 2 - 7 + 5)x + (-1 + 1/6 - 3/2 + 7/3 - 5/6)x^2 + (1/4 - 7/36 + 5/108)x^3$$
$$I_2 = 14 - x + (-1/9)x^2 + (1/27)x^3$$

Step 6 — Outer Integral: Integrate with Respect to x

Finally, we integrate I_2 from x = 0 to x = 6:

$$\int_0^6 [14 - x - x^2/9 + x^3/27]\ dx$$
$$= [14x - x^2/2 - x^3/27 + x^4/108]_0^6$$

Evaluating at x = 6:

$$= 14(6) - (6)^2/2 - (6)^3/27 + (6)^4/108$$
$$= 84 - 18 - 216/27 + 1296/108$$
$$= 84 - 18 - 8 + 12$$
$$\mathbf{= 70}$$

5. Final Answer

$$\iiint V\ (x + y + z)\ dV \ = \ 70$$

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6. Summary of Limits and Integral Structure

Variable	Lower Limit	Upper Limit
z	0	$3 - x/2 - 3y/4$
y	0	$4 - 2x/3$
x	0	6

Solution method: Cartesian coordinates | Integration order: dz dy dx