# Chapter 1: Differential Calculus

Lecture Notes

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## Unit 1: Lecture Notes

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## Topic 1: Mean Value Theorems and Applications

#### Introduction

Mean Value Theorems are fundamental results in calculus that establish relationships between the values of a function and its derivatives. These theorems provide the theoretical foundation for many results in calculus and have important applications in analyzing function behavior.

#### 1.1 Rolle's Theorem

**Statement:** Let $f(x)$ be a function satisfying: 1. $f(x)$ is continuous on the closed interval $[a, b]$ 2. $f(x)$ is differentiable on the open interval $(a, b)$ 3. $f(a) = f(b)$

Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

**Geometric Interpretation:** If a continuous and differentiable curve starts and ends at the same height, there must be at least one point where the tangent is horizontal.

**Detailed Example:**

**Problem:** Verify Rolle's theorem for $f(x) = x^2 - 5x + 6$ on $[2, 3]$ and find the value of $c$.

**Solution:**

**Step 1:** Check continuity on $[2, 3]$

Since $f(x) = x^2 - 5x + 6$ is a polynomial, it is continuous everywhere, including on $[2, 3]$.

**Step 2:** Check differentiability on $(2, 3)$

$f'(x) = 2x - 5$

This derivative exists for all $x$, so $f(x)$ is differentiable on $(2, 3)$.

**Step 3:** Check if $f(a) = f(b)$

$f(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0$

$f(3) = (3)^2 - 5(3) + 6 = 9 - 15 + 6 = 0$

Since $f(2) = f(3) = 0$, the condition is satisfied.

**Step 4:** Find $c$ such that $f'(c) = 0$

$f'(c) = 2c - 5 = 0$

$2c = 5$

$c = \frac{5}{2} = 2.5$

**Step 5:** Verify $c \in (2, 3)$

Since $2 < 2.5 < 3$, we have $c \in (2, 3)$.

**Conclusion:** All conditions of Rolle's theorem are satisfied, and $c = 2.5$ is the point where the tangent is horizontal.

### 1.2 Lagrange's Mean Value Theorem (LMVT)

**Statement:** Let $f(x)$ be a function satisfying: 1. $f(x)$ is continuous on $[a, b]$ 2. $f(x)$ is differentiable on $(a, b)$

Then there exists at least one point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

**Geometric Interpretation:** The instantaneous rate of change (derivative) at some point equals the average rate of change over the interval.

**Note:** Rolle's theorem is a special case of LMVT where $f(a) = f(b)$, making the right side zero.

**Detailed Example:**

**Problem:** Verify Lagrange's Mean Value Theorem for $f(x) = x^3 - 2x^2 + x + 3$ on $[0, 2]$ and find all values of $c$.

**Solution:**

**Step 1:** Check continuity on $[0, 2]$

$f(x) = x^3 - 2x^2 + x + 3$ is a polynomial, hence continuous on $[0, 2]$.

**Step 2:** Check differentiability on $(0, 2)$

$f'(x) = 3x^2 - 4x + 1$

This exists for all $x$, so $f(x)$ is differentiable on $(0, 2)$.

**Step 3:** Calculate $f(a)$ and $f(b)$

$f(0) = (0)^3 - 2(0)^2 + 0 + 3 = 3$

$f(2) = (2)^3 - 2(2)^2 + 2 + 3 = 8 - 8 + 2 + 3 = 5$

**Step 4:** Calculate the average rate of change

$$\frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(0)}{2 - 0} = \frac{5 - 3}{2} = \frac{2}{2} = 1$$

**Step 5:** Find $c$ such that $f'(c) = 1$

$$f'(c) = 3c^2 - 4c + 1 = 1$$

$$3c^2 - 4c + 1 - 1 = 0$$

$$3c^2 - 4c = 0$$

$$c(3c - 4) = 0$$

$$c = 0 \quad \text{or} \quad c = \frac{4}{3}$$

**Step 6:** Check which values lie in $(0, 2)$

- $c = 0$ is not in the open interval $(0, 2)$
- $c = \frac{4}{3} \approx 1.333$ is in $(0, 2)$

**Conclusion:** Lagrange's MVT is verified, and $c = \frac{4}{3}$ is the point where the instantaneous rate equals the average rate.

### 1.3 Cauchy's Mean Value Theorem (Generalized MVT)

**Statement:** Let $f(x)$ and $g(x)$ be two functions satisfying: 1. Both are continuous on $[a, b]$ 2. Both are differentiable on $(a, b)$ 3. $g'(x) \neq 0$ for all $x \in (a, b)$

Then there exists at least one point $c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

**Note:** When $g(x) = x$, Cauchy's theorem reduces to Lagrange's MVT.

**Detailed Example:**

**Problem:** Verify Cauchy's Mean Value Theorem for $f(x) = x^2$ and $g(x) = x^3$ on $[1, 2]$ and find the value of $c$.

**Solution:**

**Step 1:** Check continuity on $[1, 2]$

Both $f(x) = x^2$ and $g(x) = x^3$ are polynomials, hence continuous on $[1, 2]$.

**Step 2:** Check differentiability on $(1, 2)$

$f'(x) = 2x$ — exists for all $x$

$g'(x) = 3x^2$ — exists for all $x$

**Step 3:** Verify $g'(x) \neq 0$ on $(1, 2)$

$g'(x) = 3x^2 \neq 0$ for any $x \in (1, 2)$ since $x > 0$

**Step 4:** Calculate function values

$f(1) = (1)^2 = 1$, $f(2) = (2)^2 = 4$

$g(1) = (1)^3 = 1$, $g(2) = (2)^3 = 8$

**Step 5:** Calculate the right-hand side of Cauchy's formula

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(2) - f(1)}{g(2) - g(1)} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$

**Step 6:** Find $c$ such that $\frac{f'(c)}{g'(c)} = \frac{3}{7}$

$$\frac{f'(c)}{g'(c)} = \frac{2c}{3c^2} = \frac{2}{3c}$$

Setting this equal to $\frac{3}{7}$:

$$\frac{2}{3c} = \frac{3}{7}$$

$$2 \times 7 = 3 \times 3c$$

$$14 = 9c$$

$$c = \frac{14}{9} \approx 1.556$$

**Step 7:** Verify $c \in (1, 2)$

Since $1 < \frac{14}{9} < 2$, we have $c \in (1, 2)$.

**Conclusion:** Cauchy's Mean Value Theorem is verified with $c = \frac{14}{9}$.

#### Applications of Mean Value Theorems

**Application 1:** Proving inequalities

**Application 2:** Showing that a function is constant if its derivative is zero

**Application 3:** Approximating function values

**Application 4:** Proving existence of roots

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## Topic 2: Indeterminate Forms - Removed from Syllabus

#### Introduction

When evaluating limits, we sometimes encounter expressions whose value cannot be determined directly. These are called **indeterminate forms**. The most common indeterminate forms are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \times \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

**L'Hôpital's Rule** is the primary tool for evaluating limits of indeterminate forms.

#### L'Hôpital's Rule

**For $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms:**

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

**Indeterminate Form 1:** $\frac{0}{0}$

**Detailed Example:**

**Problem:** Evaluate $\lim_{x\to 0} \frac{\sin 3x}{2x}$

**Solution:**

**Step 1:** Check if it's an indeterminate form

$$\lim_{x\to 0} \sin 3x = \sin 0 = 0$$

$$\lim_{x\to 0} 2x = 0$$

This gives $\frac{0}{0}$, which is indeterminate.

**Step 2:** Apply L'Hôpital's Rule

$$\lim_{x\to 0} \frac{\sin 3x}{2x} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sin 3x)}{\frac{d}{dx}(2x)}$$

**Step 3:** Differentiate numerator and denominator

Numerator: $\frac{d}{dx}(\sin 3x) = 3\cos 3x$

Denominator: $\frac{d}{dx}(2x) = 2$

**Step 4:** Evaluate the new limit

$$\lim_{x\to 0} \frac{3\cos 3x}{2} = \frac{3\cos 0}{2} = \frac{3\times 1}{2} = \frac{3}{2}$$

**Answer:** $\frac{3}{2}$

---

**Indeterminate Form 2:** $\frac{\infty}{\infty}$

**Detailed Example:**

**Problem:** Evaluate $\lim_{x\to\infty} \frac{x^2+3x}{2x^2-5}$

**Solution:**

**Step 1:** Check if it's an indeterminate form

As $x \to \infty$: numerator $\to \infty$, denominator $\to \infty$

This gives $\frac{\infty}{\infty}$, which is indeterminate.

**Step 2:** Apply L'Hôpital's Rule

$$\lim_{x\to\infty} \frac{x^2+3x}{2x^2-5} = \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2+3x)}{\frac{d}{dx}(2x^2-5)}$$

**Step 3:** Differentiate

Numerator: $\frac{d}{dx}(x^2+3x) = 2x+3$

Denominator: $\frac{d}{dx}(2x^2-5) = 4x$

**Step 4:** Evaluate the new limit

$$\lim_{x\to\infty} \frac{2x+3}{4x}$$

This is still $\frac{\infty}{\infty}$, so apply L'Hôpital's Rule again:

$$\lim_{x\to\infty} \frac{\frac{d}{dx}(2x+3)}{\frac{d}{dx}(4x)} = \lim_{x\to\infty} \frac{2}{4} = \frac{1}{2}$$

**Answer:** $\frac{1}{2}$

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**Indeterminate Form 3:** $0 \times \infty$

**Strategy:** Convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form by rewriting as:

$$0 \times \infty = \frac{0}{1/\infty} = \frac{0}{0} \quad \text{or} \quad 0 \times \infty = \frac{\infty}{1/0} = \frac{\infty}{\infty}$$

**Detailed Example:**

**Problem:** Evaluate $\lim_{x\to 0^+} x \ln x$

**Solution:**

**Step 1:** Identify the form

As $x \to 0^+$: $x \to 0$ and $\ln x \to -\infty$

This gives $0 \times (-\infty)$, which is indeterminate.

**Step 2:** Rewrite to get $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$$x \ln x = \frac{\ln x}{1/x}$$

Now as $x \to 0^+$: $\ln x \to -\infty$ and $\frac{1}{x} \to +\infty$

This gives $\frac{-\infty}{\infty}$.

**Step 3:** Apply L'Hôpital's Rule

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)}$$

**Step 4:** Differentiate

Numerator: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Denominator: $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = -\frac{1}{x^2}$

**Step 5:** Simplify and evaluate

$$\lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{1}{x} \times \frac{x^2}{-1} = \lim_{x \to 0^+} \frac{x^2}{-x} = \lim_{x \to 0^+} (-x) = 0$$

**Answer:** $0$

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**Indeterminate Form 4:** $\infty - \infty$

**Strategy:** Combine terms algebraically to convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

**Detailed Example:**

**Problem:** Evaluate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

**Solution:**

**Step 1:** Identify the form

As $x \to 0$: $\frac{1}{x} \to \infty$ and $\frac{1}{\sin x} \to \infty$

This gives $\infty - \infty$, which is indeterminate.

**Step 2:** Combine into a single fraction

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x \sin x}$$

**Step 3:** Check the new form

As $x \to 0$: - Numerator: $\sin x - x \to 0 - 0 = 0$ - Denominator: $x \sin x \to 0 \times 0 = 0$

This gives $\frac{0}{0}$.

**Step 4:** Apply L'Hôpital's Rule

$$\lim_{x \to 0} \frac{\sin x - x}{x \sin x} = \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x - x)}{\frac{d}{dx}(x \sin x)}$$

**Step 5:** Differentiate

Numerator: $\frac{d}{dx}(\sin x - x) = \cos x - 1$

Denominator: $\frac{d}{dx}(x \sin x) = \sin x + x \cos x$ (product rule)

**Step 6:** Evaluate

$$\lim_{x \to 0} \frac{\cos x - 1}{\sin x + x \cos x} = \frac{\cos 0 - 1}{\sin 0 + 0 \times \cos 0} = \frac{1 - 1}{0 + 0} = \frac{0}{0}$$

Still indeterminate! Apply L'Hôpital's Rule again:

**Step 7:** Differentiate again

Numerator: $\frac{d}{dx}(\cos x - 1) = -\sin x$

Denominator: $\frac{d}{dx}(\sin x + x \cos x) = \cos x + \cos x - x \sin x = 2 \cos x - x \sin x$

**Step 8:** Evaluate

$$\lim_{x \to 0} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{-\sin 0}{2 \cos 0 - 0 \times \sin 0} = \frac{0}{2 \times 1 - 0} = \frac{0}{2} = 0$$

**Answer:** $0$

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**Indeterminate Form 5:** $1^\infty$

**Strategy:** Use the formula: If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$, then:

$$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$$

Or use: $y = [f(x)]^{g(x)} \Rightarrow \ln y = g(x)\ln f(x)$

**Detailed Example:**

**Problem:** Evaluate $\lim_{x\to 0}(1+x)^{1/x}$

**Solution:**

**Step 1:** Identify the form

As $x \to 0$: $(1+x) \to 1$ and $\frac{1}{x} \to \infty$

This gives $1^\infty$, which is indeterminate.

**Step 2:** Take the natural logarithm

Let $y = (1+x)^{1/x}$

$$\ln y = \ln[(1+x)^{1/x}] = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$$

**Step 3:** Find $\lim_{x\to 0} \ln y$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+x)}{x}$$

As $x \to 0$: $\ln(1+x) \to \ln 1 = 0$ and $x \to 0$

This gives $\frac{0}{0}$.

**Step 4:** Apply L'Hôpital's Rule

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[\ln(1+x)]}{\frac{d}{dx}(x)}$$

Numerator: $\frac{d}{dx}[\ln(1+x)] = \frac{1}{1+x}$

Denominator: $\frac{d}{dx}(x) = 1$

**Step 5:** Evaluate

$$\lim_{x \to 0} \frac{1/(1+x)}{1} = \lim_{x \to 0} \frac{1}{1+x} = \frac{1}{1+0} = 1$$

So $\lim_{x \to 0} \ln y = 1$

**Step 6:** Find the original limit

Since $\ln y \to 1$, we have:

$$y = e^{\ln y} \to e^1 = e$$

**Answer:** $e$

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**Indeterminate Form 6:** $0^0$

**Strategy:** Similar to $1^\infty$, use logarithms: $y = [f(x)]^{g(x)} \Rightarrow \ln y = g(x) \ln f(x)$

**Detailed Example:**

**Problem:** Evaluate $\lim_{x \to 0^+} x^x$

**Solution:**

**Step 1:** Identify the form

As $x \to 0^+$: base $x \to 0$ and exponent $x \to 0$

This gives $0^0$, which is indeterminate.

**Step 2:** Take logarithm

Let $y = x^x$

$$\ln y = \ln(x^x) = x \ln x$$

**Step 3:** Find $\lim_{x \to 0^+} \ln y$

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln x$$

This is the $0 \times \infty$ form (we solved this earlier!)

From our previous work: $\lim_{x\to 0^+} x \ln x = 0$

**Step 4:** Find the original limit

Since $\ln y \to 0$:

$$y = e^{\ln y} \to e^0 = 1$$

**Answer:** $1$

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**Indeterminate Form 7:** $\infty^0$

**Strategy:** Use logarithms as before.

**Detailed Example:**

**Problem:** Evaluate $\lim_{x\to\infty} x^{1/x}$

**Solution:**

**Step 1:** Identify the form

As $x \to \infty$: base $x \to \infty$ and exponent $\frac{1}{x} \to 0$

This gives $\infty^0$, which is indeterminate.

**Step 2:** Take logarithm

Let $y = x^{1/x}$

$$\ln y = \ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

**Step 3:** Find $\lim_{x\to\infty} \ln y$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x}$$

As $x \to \infty$: $\ln x \to \infty$ and $x \to \infty$

This gives $\frac{\infty}{\infty}$.

**Step 4:** Apply L'Hôpital's Rule

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x)} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0$$

**Step 5:** Find the original limit

Since $\ln y \to 0$:

$$y = e^{\ln y} \to e^0 = 1$$

**Answer:** $1$

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## Topic 3: Taylor's and Maclaurin's Series Expansion with Remainders

### Introduction

Taylor series allow us to represent functions as infinite polynomials, which is extremely useful for approximations, solving differential equations, and analyzing function behavior.

### 3.1 Taylor Series

**Definition:** If a function $f(x)$ has derivatives of all orders at $x = a$, then the **Taylor series** of $f(x)$ about $x = a$ is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

**General Form:**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ denotes the $n$-th derivative of $f$ evaluated at $x = a$.

#### 3.2 Maclaurin Series

**Definition:** A **Maclaurin series** is a special case of Taylor series where $a = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

**General Form:**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

#### 3.3 Common Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

#### 3.4 Taylor's Theorem with Remainder

When we approximate a function using only the first $n$ terms of its Taylor series, there is an error. The remainder term quantifies this error.

**Taylor's Formula with Lagrange Remainder:**

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

where the **Lagrange form of the remainder** is:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c$ between $a$ and $x$.

**For Maclaurin series** ($a = 0$):

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

for some $c$ between 0 and $x$.

#### Detailed Example: Maclaurin Series

**Problem:** Find the Maclaurin series for $f(x) = e^{2x}$ up to the term containing $x^4$, and use it to approximate $e^{0.2}$. Estimate the error using the remainder term.

**Solution:**

**Part 1: Find the Maclaurin series**

**Step 1:** Calculate derivatives and evaluate at $x = 0$

$f(x) = e^{2x}$ $f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$ $f'(0) = 2e^0 = 2$

$f''(x) = 4e^{2x}$ $f''(0) = 4e^0 = 4$

$f'''(x) = 8e^{2x}$ $f'''(0) = 8e^0 = 8$

$f^{(4)}(x) = 16e^{2x}$ $f^{(4)}(0) = 16e^0 = 16$

$f^{(5)}(x) = 32e^{2x}$ $f^{(5)}(0) = 32e^0 = 32$

**Step 2:** Write the Maclaurin series up to $x^4$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$e^{2x} = 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \frac{16}{24}x^4 + \cdots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots$$

**Part 2: Approximate $e^{0.2}$**

We want to approximate $e^{2(0.1)} = e^{0.2}$ using $x = 0.1$:

$$e^{0.2} \approx 1 + 2(0.1) + 2(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{2}{3}(0.1)^4$$

$$= 1 + 0.2 + 2(0.01) + \frac{4}{3}(0.001) + \frac{2}{3}(0.0001)$$

$$= 1 + 0.2 + 0.02 + \frac{0.004}{3} + \frac{0.0002}{3}$$

$$= 1 + 0.2 + 0.02 + 0.001333... + 0.0000667...$$

$$\approx 1.2214$$

**Part 3: Estimate the error**

The error is given by the remainder term:

$$R_4(x) = \frac{f^{(5)}(c)}{5!}x^5$$

where $c$ is between 0 and $x = 0.1$.

$$R_4(0.1) = \frac{32e^{2c}}{120}(0.1)^5 = \frac{32e^{2c}}{120}(0.00001)$$

Since $0 < c < 0.1$, we have $1 < e^{2c} < e^{0.2} < 1.25$ (approximately).

Maximum error:

$$|R_4(0.1)| < \frac{32 \times 1.25}{120} \times 0.00001 = \frac{40}{120} \times 0.00001 = \frac{1}{3} \times 0.00001 \approx 0.0000033$$

**Answer:** - Maclaurin series: $e^{2x} = 1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots$ - Approximation: $e^{0.2} \approx 1.2214$ - Maximum error: $< 0.0000034$

(Actual value: $e^{0.2} \approx 1.2214027...$, so our approximation is very accurate!)

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## Topic 4: Concavity and Convexity of a Curve

#### Definitions

**Concave Up (Convex):** A function $f(x)$ is **concave up** (or convex) on an interval if its graph lies above all its tangent lines on that interval. Geometrically, the curve "holds water."

**Concave Down (Concave):** A function $f(x)$ is **concave down** (or concave) on an interval if its graph lies below all its tangent lines on that interval. Geometrically, the curve "spills water."

#### Mathematical Test for Concavity

**Second Derivative Test:**

1. If $f''(x) > 0$ on an interval, then $f(x)$ is **concave up** on that interval.
2. If $f''(x) < 0$ on an interval, then $f(x)$ is **concave down** on that interval.

#### Detailed Example 1: Concavity

**Problem:** Determine the intervals where $f(x) = x^3 - 6x^2 + 9x + 1$ is concave up and concave down.

**Solution:**

**Step 1:** Find the first derivative

$$f'(x) = 3x^2 - 12x + 9$$

**Step 2:** Find the second derivative

$$f''(x) = 6x - 12$$

**Step 3:** Find where $f''(x) = 0$

$$6x - 12 = 0$$

$$6x = 12$$

$$x = 2$$

**Step 4:** Test the sign of $f''(x)$ on intervals divided by $x = 2$

**Interval 1:** $x < 2$ (test point: $x = 0$)

$$f''(0) = 6(0) - 12 = -12 < 0$$

So $f$ is **concave down** on $(-\infty, 2)$.

**Interval 2:** $x > 2$ (test point: $x = 3$)

$$f''(3) = 6(3) - 12 = 18 - 12 = 6 > 0$$

So $f$ is **concave up** on $(2, \infty)$.

**Answer:** - **Concave up:** $(2, \infty)$ - **Concave down:** $(-\infty, 2)$

**Detailed Example 2: Concavity**

**Problem:** Determine the concavity of $f(x) = e^{-x^2}$.

**Solution:**

**Step 1:** Find the first derivative

$$f'(x) = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$$

**Step 2:** Find the second derivative (using product rule)

$$f''(x) = \frac{d}{dx}[-2xe^{-x^2}]$$

$$= -2e^{-x^2} + (-2x)(-2x)e^{-x^2}$$

$$= -2e^{-x^2} + 4x^2e^{-x^2}$$

$$= e^{-x^2}(4x^2 - 2)$$

$$= 2e^{-x^2}(2x^2 - 1)$$

**Step 3:** Find where $f''(x) = 0$

Since $e^{-x^2} > 0$ for all $x$:

$$\begin{gathered} 2x^2 - 1 = 0 \\ x^2 = \frac{1}{2} \\ x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \end{gathered}$$

**Step 4:** Test the sign of $f''(x)$

**Interval 1:** $x < -\frac{\sqrt{2}}{2}$ (test: $x = -1$)

$$f''(-1) = 2e^{-1}(2(1) - 1) = 2e^{-1}(1) > 0$$

**Concave up**

**Interval 2:** $-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$ (test: $x = 0$)

$$f''(0) = 2e^{0}(2(0) - 1) = 2(1)(-1) = -2 < 0$$

**Concave down**

**Interval 3:** $x > \frac{\sqrt{2}}{2}$ (test: $x = 1$)

$$f''(1) = 2e^{-1}(2(1) - 1) = 2e^{-1}(1) > 0$$

**Concave up**

**Answer: - Concave up:** $\left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, \infty\right)$ **- Concave down:** $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

---

## Topic 5: Points of Inflection

#### Definition

A **point of inflection** is a point on the curve where the concavity changes (from concave up to concave down, or vice versa).

#### Finding Points of Inflection

**Steps:** 1. Find $f''(x)$ 2. Solve $f''(x) = 0$ or find where $f''(x)$ is undefined 3. Check if the concavity actually changes at these points 4. If concavity changes, $(c, f(c))$ is a point of inflection

**Note:** $f''(c) = 0$ is necessary but not sufficient. The second derivative must change sign.

#### Detailed Example 1: Points of Inflection

**Problem:** Find all points of inflection for $f(x) = x^4 - 4x^3$.

**Solution:**

**Step 1:** Find the second derivative

$$f'(x) = 4x^3 - 12x^2$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

**Step 2:** Solve $f''(x) = 0$

$$12x(x-2) = 0$$

$$x = 0 \quad \text{or} \quad x = 2$$

**Step 3:** Check if concavity changes at these points

**At $x = 0$:**

Test $x = -1$: $f''(-1) = 12(-1)(-1-2) = 12(-1)(-3) = 36 > 0$ (concave up)

Test $x = 1$: $f''(1) = 12(1)(1-2) = 12(1)(-1) = -12 < 0$ (concave down)

Concavity changes from up to down **Point of inflection**

**At $x = 2$:**

Test $x = 1$: $f''(1) = -12 < 0$ (concave down)

Test $x = 3$: $f''(3) = 12(3)(3 - 2) = 12(3)(1) = 36 > 0$ (concave up)

Concavity changes from down to up **Point of inflection**

**Step 4:** Find the y-coordinates

$$f(0) = 0^4 - 4(0)^3 = 0$$

$$f(2) = 2^4 - 4(2)^3 = 16 - 32 = -16$$

**Answer:** Points of inflection are $(0, 0)$ and $(2, -16)$.

#### Detailed Example 2: Points of Inflection

**Problem:** Find all points of inflection for $f(x) = \sin x$ on $[0, 2\pi]$.

**Solution:**

**Step 1:** Find the second derivative

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

**Step 2:** Solve $f''(x) = 0$

$$-\sin x = 0$$
$$\sin x = 0$$

On $[0, 2\pi]$: $x = 0, \pi, 2\pi$

**Step 3:** Check if concavity changes

**At $x = 0$:**

Test $x = -0.1$ (slightly left): $f''(-0.1) = -\sin(-0.1) \approx 0.0998 > 0$

Test $x = 0.1$ (slightly right): $f''(0.1) = -\sin(0.1) \approx -0.0998 < 0$

Concavity changes **Point of inflection at** $x = 0$ (boundary point)

**At** $x = \pi$**:**

Test $x = 3$ (slightly left, since $\pi \approx 3.14$): $f''(3) = -\sin(3) \approx -0.141 < 0$

Test $x = 3.5$ (slightly right): $f''(3.5) = -\sin(3.5) \approx 0.351 > 0$

Concavity changes **Point of inflection**

**At** $x = 2\pi$**:**

Test $x = 6$ (slightly left, since $2\pi \approx 6.28$): $f''(6) = -\sin(6) \approx 0.279 > 0$

Test $x = 6.5$ (slightly right): $f''(6.5) = -\sin(6.5) \approx -0.215 < 0$

Concavity changes **Point of inflection at** $x = 2\pi$ (boundary point)

**Step 4:** Find y-coordinates

$$f(0) = \sin(0) = 0$$

$$f(\pi) = \sin(\pi) = 0$$

$$f(2\pi) = \sin(2\pi) = 0$$

**Answer:** Points of inflection are $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$.

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## Topic 6: Curve Sketching

### Introduction

Curve sketching involves analyzing a function systematically to draw its graph accurately. We use derivatives and other tools to determine key features.

### Steps for Curve Sketching

**Step 1: Domain** - Find the domain of $f(x)$ (where is it defined?)

**Step 2: Intercepts** - **y-intercept:** Set $x = 0$, find $f(0)$ - **x-intercepts:** Set $f(x) = 0$, solve for $x$

**Step 3: Symmetry** - **Even function:** $f(-x) = f(x)$ (symmetric about y-axis) - **Odd function:** $f(-x) = -f(x)$ (symmetric about origin)

**Step 4: Asymptotes** - **Vertical asymptotes:** Where denominator = 0 (for rational functions) - **Horizontal asymptotes:** $\lim_{x\to\pm\infty} f(x)$ - **Oblique asymptotes:** If degree of numerator = degree of denominator + 1

**Step 5: First Derivative Analysis** - Find $f'(x)$ - Find critical points: Solve $f'(x) = 0$ or where $f'(x)$ is undefined - Determine intervals of increase/decrease - Identify local maxima and minima

**Step 6: Second Derivative Analysis** - Find $f''(x)$ - Determine concavity: Where is $f''(x) > 0$ or $f''(x) < 0$? - Find inflection points: Where does $f''(x) = 0$ and concavity changes?

**Step 7: Sketch the Curve** - Plot all key points (intercepts, extrema, inflection points) - Draw asymptotes - Connect points smoothly according to increase/decrease and concavity

#### Detailed Example 1: Complete Curve Sketching

**Problem:** Sketch the curve $f(x) = \frac{x^2}{x-1}$.

**Solution:**

**Step 1: Domain**

The function is undefined when $x - 1 = 0$, i.e., $x = 1$.

**Domain:** $(-\infty, 1) \cup (1, \infty)$

**Step 2: Intercepts**

**y-intercept:** $f(0) = \frac{0^2}{0-1} = \frac{0}{-1} = 0$ Point: $(0, 0)$

**x-intercepts:** Set $f(x) = 0$

$$\frac{x^2}{x-1} = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$$

Point: $(0, 0)$

**Step 3: Symmetry**

$$f(-x) = \frac{(-x)^2}{-x-1} = \frac{x^2}{-x-1}$$

This is neither $f(x)$ nor $-f(x)$, so **no symmetry**.

**Step 4: Asymptotes**

**Vertical asymptote:** At $x = 1$ (where denominator = 0)

Check behavior near $x = 1$: - As $x \to 1^-$: numerator $\to 1$, denominator $\to 0^-$, so $f(x) \to -\infty$ - As $x \to 1^+$: numerator $\to 1$, denominator $\to 0^+$, so $f(x) \to +\infty$

**Horizontal asymptote:**

$$\lim_{x\to\infty} \frac{x^2}{x-1}$$

Degree of numerator (2) > degree of denominator (1), so **no horizontal asymptote**.

**Oblique asymptote:**

Perform polynomial long division:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

As $x \to \pm\infty$, $\frac{1}{x-1} \to 0$, so the oblique asymptote is $y = x + 1$.

**Step 5: First Derivative Analysis**

Using quotient rule:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

$$f'(x) = \frac{x(x-2)}{(x-1)^2}$$

**Critical points:** $f'(x) = 0$

$$x(x-2) = 0 \Rightarrow x = 0 \text{ or } x = 2$$

**Sign analysis of $f'(x)$:**

| Interval | $(-\infty, 0)$ | $(0, 1)$ | $(1, 2)$ | $(2, \infty)$ |
|---|---|---|---|---|
| $x$ | $-$ | $+$ | $+$ | $+$ |
| $x - 2$ | $-$ | $-$ | $-$ | $+$ |
| $(x-1)^2$ | $+$ | $+$ | $+$ | $+$ |
| $f'(x)$ | $+$ | $-$ | $-$ | $+$ |

| Interval | $(-\infty, 0)$ | $(0, 1)$ | $(1, 2)$ | $(2, \infty)$ |
|---|---|---|---|---|
| $f(x)$ | Increasing | Decreasing | Decreasing | Increasing |

**Local maximum at $x = 0$:** $f(0) = 0$, point $(0, 0)$

**Local minimum at $x = 2$:** $f(2) = \frac{4}{1} = 4$, point $(2, 4)$

**Step 6: Second Derivative Analysis**

This gets complex. Let's use the quotient rule on $f'(x) = \frac{x^2-2x}{(x-1)^2}$:

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x) \cdot 2(x-1)}{(x-1)^4}$$

$$= \frac{(x-1)[(x-1)(2x-2) - 2(x^2-2x)]}{(x-1)^4}$$

$$= \frac{(x-1)(2x-2) - 2(x^2-2x)}{(x-1)^3}$$

$$= \frac{2x^2 - 2x - 2x + 2 - 2x^2 + 4x}{(x-1)^3}$$

$$= \frac{2}{(x-1)^3}$$

**Concavity:**

$f''(x) = \frac{2}{(x-1)^3}$

- For $x < 1$: $(x-1)^3 < 0$, so $f''(x) < 0$ → **concave down**
- For $x > 1$: $(x-1)^3 > 0$, so $f''(x) > 0$ → **concave up**

**Inflection points:** $f''(x)$ is never zero, only undefined at $x = 1$ (which is not in domain).

**No inflection points.**

**Step 7: Sketch**

**Summary of key features:** - Domain: $(-\infty, 1) \cup (1, \infty)$ - Intercepts: $(0, 0)$ - Vertical asymptote: $x = 1$ - Oblique asymptote: $y = x + 1$ - Local max: $(0, 0)$ - Local min: $(2, 4)$ - Increasing on: $(-\infty, 0) \cup (2, \infty)$ - Decreasing on: $(0, 1) \cup (1, 2)$ - Concave down: $(-\infty, 1)$ - Concave up: $(1, \infty)$

**Sketch:**

```
        |     /
        |    / y = x+1 (oblique asymptote)
     4  + (2,4) min
        |  /
        | /
   -----+/-----(0,0)max----- x
       /|
      / |
     /  | x=1 (vertical asymptote)
    /   |
```

#### Detailed Example 2: Complete Curve Sketching

**Problem:** Sketch the curve $f(x) = x^3 - 3x^2 + 2$.

**Solution:**

**Step 1: Domain**

Polynomial function, defined everywhere.

**Domain:** $(-\infty, \infty)$

**Step 2: Intercepts**

**y-intercept:** $f(0) = 0 - 0 + 2 = 2$ Point: $(0, 2)$

**x-intercepts:** Set $f(x) = 0$

$$x^3 - 3x^2 + 2 = 0$$

Try $x = 1$: $1 - 3 + 2 = 0$

Factor: $(x - 1)(x^2 - 2x - 2) = 0$

Using quadratic formula on $x^2 - 2x - 2 = 0$:

$$x = \frac{2 \pm \sqrt{4 + 8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

**x-intercepts:** $x = 1, x = 1 + \sqrt{3} \approx 2.73, x = 1 - \sqrt{3} \approx -0.73$

**Step 3: Symmetry**

$$f(-x) = (-x)^3 - 3(-x)^2 + 2 = -x^3 - 3x^2 + 2$$

Not equal to $f(x)$ or $-f(x)$, so **no symmetry**.

**Step 4: Asymptotes**

Polynomial → **no asymptotes**

End behavior: As $x \to \infty$, $f(x) \to \infty$; as $x \to -\infty$, $f(x) \to -\infty$

**Step 5: First Derivative Analysis**

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

**Critical points:** $f'(x) = 0$

$$3x(x-2) = 0 \Rightarrow x = 0 \text{ or } x = 2$$

**Sign analysis:**

| Interval | $(-\infty, 0)$ | $(0, 2)$ | $(2, \infty)$ |
|---|---|---|---|
| $f'(x)$ | $+$ | $-$ | $+$ |
| $f(x)$ | Increasing | Decreasing | Increasing |

**Local maximum at $x = 0$:** $f(0) = 2$, point $(0, 2)$

**Local minimum at $x = 2$:** $f(2) = 8 - 12 + 2 = -2$, point $(2, -2)$

**Step 6: Second Derivative Analysis**

$$f''(x) = 6x - 6 = 6(x-1)$$

**Find where $f''(x) = 0$:**

$$6(x-1) = 0 \Rightarrow x = 1$$

**Concavity:**

| Interval | $(-\infty, 1)$ | $(1, \infty)$ |
|---|---|---|
| $f''(x)$ | $-$ | $+$ |
| Concavity | Down | Up |

**Inflection point at $x = 1$:**

$$f(1) = 1 - 3 + 2 = 0$$

Point: $(1, 0)$

**Step 7: Sketch**

**Summary:** - Domain: All real numbers - y-intercept: $(0, 2)$ - x-intercepts: $(-0.73, 0)$, $(1, 0)$, $(2.73, 0)$ - Local max: $(0, 2)$ - Local min: $(2, -2)$ - Inflection point: $(1, 0)$ - Increasing: $(-\infty, 0) \cup (2, \infty)$ - Decreasing: $(0, 2)$ - Concave down: $(-\infty, 1)$ - Concave up: $(1, \infty)$

**Sketch:**

```
     |
  2 + (0,2) max
     |    __
     |   /  \
---+--*----*---*--- x
   -0.73  1  2  2.73
     |  (1,0)  \
     |  inflec  \
-2 +          (2,-2) min
```

The curve starts from bottom left, crosses x-axis at $-0.73$, reaches maximum at $(0, 2)$, passes through inflection point at $(1, 0)$ where concavity changes, reaches minimum at $(2, -2)$, crosses x-axis at $2.73$, and continues to top right.

---

## Detailed Example 3: Sketching $f(x) = e^{-x^2}$

**Step 1: Domain**
Exponential function defined for all real numbers.
**Domain:** $(-\infty, \infty)$

**Step 2: Intercepts**
**y-intercept:** $f(0) = e^0 = 1$ → $(0, 1)$
**x-intercepts:** $e^{-x^2} > 0$ for all $x$ → **No x-intercepts**

**Step 3: Symmetry**
$f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$
Yes… Even function → Symmetric about y-axis

**Step 4: Asymptotes**
As $x \to \pm\infty$, $e^{-x^2} \to 0$
Yes… Horizontal asymptote: $y = 0$

**Step 5: First Derivative**

$$f'(x) = -2xe^{-x^2}$$

Critical point:

$$-2xe^{-x^2} = 0 \Rightarrow x = 0$$

| Interval | $(-\infty, 0)$ | $(0, \infty)$ |
|---|---|---|
| $f'(x)$ | + | - |
| Behavior | Increasing | Decreasing |

Yes… Global maximum at $(0, 1)$

**Step 6: Second Derivative**

$$f''(x) = 2e^{-x^2}(2x^2 - 1)$$

Inflection points:

$$2x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Concave down: $|x| < 1/\sqrt{2}$
Concave up: $|x| > 1/\sqrt{2}$

**Summary** - Bell-shaped curve - Symmetric - Maximum at (0,1) - Horizontal asymptote $y = 0$

---

## Detailed Example 4: Sketching $f(x) = \sin x + 2\cos x$

**Step 1: Domain**
Defined for all real numbers.

**Step 2: Convert to Amplitude Form**

$$f(x) = R\sin(x+\alpha)$$

$$R = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Let

$$\cos\alpha = \frac{1}{\sqrt{5}}, \quad \sin\alpha = \frac{2}{\sqrt{5}}$$

So:

$$f(x) = \sqrt{5}\sin(x+\alpha)$$

**Step 3: Amplitude & Period**

Amplitude: $\sqrt{5}$
Period: $2\pi$

Range: $[-\sqrt{5}, \sqrt{5}]$

**Step 4: Critical Points**

Max value: $\sqrt{5}$
Min value: $-\sqrt{5}$

Occurs when $\sin(x+\alpha) = \pm 1$

**Summary** - Periodic wave - Amplitude $\sqrt{5}$ - Phase shift $-\alpha$

---

## Detailed Example 5: Sketching $f(x) = \log(x^2)$

**Step 1: Domain**

$x^2 > 0 \Rightarrow x \neq 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

**Step 2: Rewrite**

$$\log(x^2) = 2\log|x|$$

**Step 3: Symmetry**

$f(-x) = \log((-x)^2) = \log(x^2) = f(x)$
Yes… Even function

**Step 4: Asymptotes**

As $x \to 0$, $\log(x^2) \to -\infty$
Yes… Vertical asymptote: $x = 0$

**Step 5: Derivative**

$$f'(x) = \frac{2}{x}$$

| Interval | $(-\infty, 0)$ | $(0, \infty)$ |
|---|---|---|
| $f'(x)$ | - | + |
| Behavior | Decreasing | Increasing |

**Summary** - Two symmetric branches - Vertical asymptote at $x = 0$ - No horizontal asymptote

---

## Detailed Example 6: Polar Curve $r = 1 + \cos\theta$

**Step 1: Type of Curve**

Cardioid

**Step 2: Symmetry**

Replace $\theta$ by $-\theta$:

$$r = 1 + \cos(-\theta) = 1 + \cos\theta$$

Yes… Symmetric about polar axis

**Step 3: Key Points**

| $\theta$ | $r$ |
|---|---|
| $0$ | $2$ |
| $\pi$ | $0$ |
| $\pi/2$ | $1$ |
| $3\pi/2$ | $1$ |

**Step 4: Behavior**

- Maximum radius = 2
- Touches pole at $\theta = \pi$
- Heart-shaped curve

---

# Summary

These six topics form the foundation of analyzing and understanding the behavior of functions using calculus:

1. **Mean Value Theorems** establish fundamental relationships between function values and derivatives
2. **Indeterminate Forms** provide techniques for evaluating challenging limits
3. **Taylor/Maclaurin Series** allow polynomial approximations of functions
4. **Concavity** describes how curves bend
5. **Inflection Points** mark where curves change their bending direction
6. **Curve Sketching** synthesizes all these tools to visualize functions completely

These concepts are essential for advanced calculus, optimization, numerical methods, and applications in engineering and physics.