Chapter 1: Differential Calculus

Tutorial

Dr. D Bhanu Prakash

Unit 1: Tutorial

Author: Dr. D Bhanu Prakash

Vignan's Foundation for Science, Technology and Research
Vadlamudi - 522 213, Guntur, Andhra Pradesh, India.

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Contents

1. **Mean Value Theorems and Applications** - Foundation for understanding rate of change and function behavior
2. **Indeterminate Forms** - Practical application of derivatives (L'Hôpital's Rule builds on MVT)
3. **Taylor's and Maclaurin's Series Expansion with Remainders** - Powerful approximation tool requiring solid derivative foundation
4. **Concavity and Convexity of a Curve** - Second derivative applications
5. **Points of Inflection** - Natural extension of concavity concepts
6. **Curve Sketching** - Synthesis of all previous topics

Problem Sets

1. Mean Value Theorems and Applications

Problem 1.1: Verify Rolle's Theorem for the function $f(x) = x^3 - 3x^2 + 2x$ on the interval $[0, 2]$. Find all values of c in $(0, 2)$ where $f'(c) = 0$.

Problem 1.2: Show that the equation $x^3 + x - 1 = 0$ has exactly one real root using the Mean Value Theorem.

Problem 1.3: A car travels 100 km in 2 hours. Using the Mean Value Theorem, prove that at some instant during the journey, the car's speedometer must have read exactly 50 km/h.

Problem 1.4: Let $f(x) = \sin x$. Use the Mean Value Theorem to show that $|\sin a - \sin b| \leq |a - b|$ for all real numbers a and b.

Problem 1.5: A tech company's server temperature is modeled by $T(t) = 20 + 15 \sin\left(\frac{\pi t}{12}\right)$ degrees Celsius, where t is time in hours ($0 \leq t \leq 24$). The cooling system triggers when the rate of temperature change exceeds 3°C per hour. Use the Mean Value Theorem to determine if there exists a time interval where the cooling system must activate.

2. Indeterminate Forms

Problem 2.1: Evaluate $\lim_{x \to 0} \frac{\sin 3x}{x}$ using L'Hôpital's Rule.

Problem 2.2: Find $\lim_{x \to \infty} \frac{x^3 + 2x^2 - 5}{2x^3 - x + 1}$.

Problem 2.3: Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

Problem 2.4: Determine $\lim_{x \to 0^+} x^x$ (hint: consider the form 0^0 by taking logarithms).

Problem 2.5: In signal processing, the sinc function is defined as $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ for $x \neq 0$. Find $\lim_{x \to 0} \text{sinc}(x)$ to determine the appropriate value for continuity at $x = 0$. This value is critical for digital filter design in telecommunications.

3. Taylor's and Maclaurin's Series Expansion with Remainders

Problem 3.1: Find the Maclaurin series expansion for $f(x) = e^{2x}$ up to the term containing x^4. Use this to approximate $e^{0.2}$.

Problem 3.2: Determine the Taylor series expansion of $f(x) = \ln x$ about $x = 1$ up to the fourth-degree term.

Problem 3.3: Find the Maclaurin series for $f(x) = \cos x$ and use the Lagrange remainder formula to estimate the error when approximating $\cos(0.5)$ using the first three non-zero terms.

Problem 3.4: Show that the Maclaurin series for $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$ and determine its radius of convergence.

Problem 3.5: In computer graphics, the rotation matrix uses $\sin\theta$ and $\cos\theta$. For small angles (in radians), programmers often use approximations. Derive the Maclaurin series for $\sin\theta$ up to θ^5 and estimate the maximum error when using only $\sin\theta \approx \theta - \frac{\theta^3}{6}$ for $|\theta| \leq 0.1$. This affects rendering accuracy in 3D game engines.

4. Concavity and Convexity of a Curve

Problem 4.1: Determine the intervals where $f(x) = x^4 - 4x^3 + 6$ is concave up and concave down.

Problem 4.2: Show that $f(x) = e^x$ is convex on $(-\infty, \infty)$.

Problem 4.3: For what values of a is the function $f(x) = ax^2 + \ln x$ concave down on $(0, \infty)$?

Problem 4.4: Prove that $f(x) = \sin x$ is concave down on $(0, \pi)$ and concave up on $(\pi, 2\pi)$.

Problem 4.5: A machine learning model's loss function is $L(\theta) = \theta^4 - 8\theta^3 + 18\theta^2 + 5$, where θ represents a model parameter. Determine where $L(\theta)$ is convex (concave up). Convexity is crucial because gradient descent optimization algorithms are guaranteed to find the global minimum only in convex regions.

5. Points of Inflection

Problem 5.1: Find all points of inflection for $f(x) = x^3 - 6x^2 + 9x + 1$.

Problem 5.2: Determine whether $f(x) = x^4$ has any points of inflection.

Problem 5.3: Show that $f(x) = x + \sin x$ has infinitely many points of inflection and find their general form.

Problem 5.4: For $f(x) = \frac{x}{x^2+1}$, find all points of inflection and verify them using the second derivative test.

Problem 5.5: A pandemic spread model uses the logistic function $P(t) = \frac{10000}{1+99e^{-0.5t}}$ for the number of infected individuals at time t (days). Find the inflection point of this curve. This point is epidemiologically significant as it indicates when the infection rate transitions from accelerating to decelerating—the optimal time for public health interventions.

6. Curve Sketching

Problem 6.1: Sketch the curve $y = \frac{x^2}{x-1}$ by finding: domain, intercepts, asymptotes, critical points, intervals of increase/decrease, concavity, and inflection points.

Problem 6.2: Sketch $f(x) = x^3 - 3x^2 + 2$ using complete analysis (critical points, extrema, concavity, inflection points).

Problem 6.3: Analyze and sketch $y = xe^{-x}$ on $(-\infty, \infty)$.

Problem 6.4: Sketch the curve $y = \frac{x}{\sqrt{x^2+1}}$ with complete analysis including horizontal asymptotes.

Problem 6.5: An engineering firm designs a suspension bridge cable modeled by the catenary function $y = 10\cosh\left(\frac{x}{10}\right) = 10 \cdot \frac{e^{x/10}+e^{-x/10}}{2}$ for $-20 \le x \le 20$ meters. Perform a complete curve sketch analysis including: minimum point (lowest cable position), concavity (cable tension distribution), and rate of climb at the anchor points ($x = \pm 20$). These determine material stress and structural safety requirements.

Solutions: Mean Value Theorems and Applications

Problem 1.1

Verify Rolle's Theorem for the function $f(x) = x^3 - 3x^2 + 2x$ on the interval $[0, 2]$. Find all values of c in $(0, 2)$ where $f'(c) = 0$.

Solution

Rolle's Theorem requires three conditions: 1. f is continuous on $[a, b]$ 2. f is differentiable on (a, b) 3. $f(a) = f(b)$

If all three hold, then there exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Step 1: Check continuity on $[0, 2]$

$f(x) = x^3 - 3x^2 + 2x$ is a polynomial. Polynomials are continuous everywhere. Yes….

Step 2: Check differentiability on $(0, 2)$

$$f'(x) = 3x^2 - 6x + 2$$

This derivative exists for all x, so f is differentiable on $(0, 2)$. Yes….

Step 3: Check $f(0) = f(2)$

$$f(0) = (0)^3 - 3(0)^2 + 2(0) = 0$$

$$f(2) = (2)^3 - 3(2)^2 + 2(2) = 8 - 12 + 4 = 0$$

Since $f(0) = f(2) = 0$, all three conditions are satisfied. Yes….

Step 4: Find all $c \in (0, 2)$ **where** $f'(c) = 0$

Set $f'(c) = 0$:

$$3c^2 - 6c + 2 = 0$$

Using the quadratic formula:

$$c = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm \sqrt{12}}{6} = \frac{6 \pm 2\sqrt{3}}{6} = 1 \pm \frac{\sqrt{3}}{3} = 1 \pm \frac{1}{\sqrt{3}}$$

So:

$$c_1 = 1 - \frac{1}{\sqrt{3}} \approx 1 - 0.577 = 0.423$$

$$c_2 = 1 + \frac{1}{\sqrt{3}} \approx 1 + 0.577 = 1.577$$

Step 5: Verify both values lie in $(0, 2)$

- $c_1 \approx 0.423 \in (0, 2)$ Yes….

- $c_2 \approx 1.577 \in (0, 2)$ Yes....

Conclusion: Rolle's Theorem is verified. There are **two** points where the tangent is horizontal:

$$\boxed{c_1 = 1 - \frac{1}{\sqrt{3}} \approx 0.423 \qquad \text{and} \qquad c_2 = 1 + \frac{1}{\sqrt{3}} \approx 1.577}$$

Problem 1.2

Show that the equation $x^3 + x - 1 = 0$ has exactly one real root using the Mean Value Theorem.

Solution

This problem has two parts: proving **existence** (at least one root) using the Intermediate Value Theorem, and proving **uniqueness** (at most one root) using Rolle's Theorem.

Part A: Existence of at least one root

Let $f(x) = x^3 + x - 1$. This is a polynomial, so it is continuous everywhere.

Evaluate at two points:

$$f(0) = 0 + 0 - 1 = -1 < 0$$

$$f(1) = 1 + 1 - 1 = 1 > 0$$

Since $f(0) < 0 < f(1)$ and f is continuous on $[0, 1]$, by the **Intermediate Value Theorem**, there exists at least one $c \in (0, 1)$ such that $f(c) = 0$.

So the equation has **at least one real root**. Yes....

Part B: Uniqueness — at most one root (proof by contradiction using Rolle's Theorem)

Suppose, for contradiction, that there are **two distinct real roots** α and β with $\alpha < \beta$.

Then $f(\alpha) = 0$ and $f(\beta) = 0$.

Since f is a polynomial: - f is continuous on $[\alpha, \beta]$ Yes…. - f is differentiable on (α, β) Yes…. - $f(\alpha) = f(\beta) = 0$ Yes….

By **Rolle's Theorem**, there must exist some $d \in (\alpha, \beta)$ such that $f'(d) = 0$.

But let us compute $f'(x)$:

$$f'(x) = 3x^2 + 1$$

Since $3x^2 \geq 0$ for all real x, we have:

$$f'(x) = 3x^2 + 1 \geq 1 > 0 \quad \text{for all } x \in \mathbb{R}$$

This means $f'(x) > 0$ everywhere — f' can **never be zero**. This contradicts the requirement that $f'(d) = 0$.

Therefore, our assumption that two distinct roots exist is **false**.

Conclusion: The equation $x^3 + x - 1 = 0$ has **exactly one real root**, located in the interval $(0, 1)$.

Exactly one real root exists, approximately $c \approx 0.6824$

Problem 1.3

A car travels 100 km in 2 hours. Using the Mean Value Theorem, prove that at some instant during the journey, the car's speedometer must have read exactly 50 km/h.

Solution

Step 1: Set up the mathematical model

Let $s(t) =$ position of the car (in km) at time t (in hours), measured from the start.

Given information: - $s(0) = 0$ (car starts at position 0) - $s(2) = 100$ (car travels 100 km in 2 hours)

The velocity of the car at any instant is $v(t) = s'(t)$.

Step 2: Verify conditions for the Mean Value Theorem

Since $s(t)$ is the position of a physical object (a car), it is: - **Continuous** on $[0, 2]$ — position cannot jump instantaneously Yes…. - **Differentiable** on $(0, 2)$ — velocity exists at every moment Yes….

Both conditions of the Mean Value Theorem are satisfied.

Step 3: Apply the Mean Value Theorem

By the **Mean Value Theorem (Lagrange)**, there exists at least one time $c \in (0, 2)$ such that:

$$s'(c) = \frac{s(2) - s(0)}{2 - 0}$$

Step 4: Compute the right-hand side

$$s'(c) = \frac{100 - 0}{2 - 0} = \frac{100}{2} = 50 \text{ km/h}$$

Conclusion: There exists at least one instant $c \in (0, 2)$ where:

$$\boxed{v(c) = s'(c) = 50 \text{ km/h}}$$

The car's instantaneous speed **must** have equalled exactly 50 km/h at some point during the 2-hour journey — regardless of how the car accelerated or braked along the way.

Physical interpretation: The average speed was 50 km/h. The MVT guarantees the speedometer must cross this value at least once, no matter how erratically the car drove.

Problem 1.4

Let $f(x) = \sin x$. Use the Mean Value Theorem to show that $|\sin a - \sin b| \leq |a - b|$ for all real numbers a and b.

Solution

Step 1: Handle the trivial case

If $a = b$, then both sides are 0:

$$|\sin a - \sin b| = 0 = |a - b|$$

So the inequality holds trivially when $a = b$.

Step 2: Apply MVT for $a \neq b$

Let $f(x) = \sin x$. Then: - f is continuous on $\mathbb{R}$, hence on $[a, b]$ (or $[b, a]$ if $b < a$) Yes.... - $f'(x) = \cos x$ exists everywhere, so f is differentiable on (a, b) Yes....

By the **Mean Value Theorem**, there exists c between a and b such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\cos c = \frac{\sin b - \sin a}{b - a}$$

Step 3: Rearrange

$$\sin b - \sin a = \cos c \cdot (b - a)$$

$$\sin a - \sin b = \cos c \cdot (a - b)$$

Step 4: Take absolute values

$$|\sin a - \sin b| = |\cos c| \cdot |a - b|$$

Step 5: Apply the key bound

It is a fundamental property that for all real c:

$$|\cos c| \leq 1$$

Therefore:

$$|\sin a - \sin b| = |\cos c| \cdot |a - b| \leq 1 \cdot |a - b|$$

$$\boxed{|\sin a - \sin b| \leq |a - b| \quad \text{for all } a, b \in \mathbb{R}}$$

Remark: This result states that $\sin x$ is a **Lipschitz continuous** function with Lipschitz constant 1. It means the sine function cannot change faster than the identity function $y = x$. This property is widely used in numerical analysis and error estimation.

Problem 1.5

A tech company's server temperature is modeled by $T(t) = 20 + 15 \sin\left(\dfrac{\pi t}{12}\right)$ **degrees Celsius, where t is time in hours ($0 \leq t \leq 24$). The cooling system triggers when the rate of temperature change exceeds 3°C per hour. Use the Mean Value Theorem to determine if there exists a time interval where the cooling system must activate.**

Solution

Step 1: Understand the cooling system condition

The cooling system activates whenever the instantaneous rate of temperature change $T'(t)$ satisfies:

$$|T'(t)| > 3°\text{C/hour}$$

We need to determine, using the MVT, whether there is guaranteed to be a time when $|T'(t)| > 3$.

Step 2: Identify a suitable time interval using the temperature function

Evaluate $T(t)$ at key points:

At $t = 0$ (midnight):
$$T(0) = 20 + 15 \sin(0) = 20°\text{C}$$

At $t = 6$ (6 AM):

$$T(6) = 20 + 15 \sin\left(\frac{\pi \cdot 6}{12}\right) = 20 + 15 \sin\left(\frac{\pi}{2}\right) = 20 + 15 = 35°C$$

Step 3: Compute the average rate of change over $[0, 6]$

$$\frac{T(6) - T(0)}{6 - 0} = \frac{35 - 20}{6} = \frac{15}{6} = 2.5°C/hour$$

This average rate is 2.5°C/hour — below 3°C/hour. Let's try another interval.

Step 4: Try the interval $[3, 6]$

At $t = 3$:

$$T(3) = 20 + 15 \sin\left(\frac{3\pi}{12}\right) = 20 + 15 \sin\left(\frac{\pi}{4}\right) = 20 + 15 \cdot \frac{\sqrt{2}}{2} = 20 + \frac{15\sqrt{2}}{2}$$

$$T(3) = 20 + \frac{15 \times 1.4142}{2} \approx 20 + 10.607 = 30.607°C$$

Average rate over $[3, 6]$:

$$\frac{T(6) - T(3)}{6 - 3} = \frac{35 - 30.607}{3} = \frac{4.393}{3} \approx 1.464°C/hour$$

Still below 3°C/hour. Let us now use a more strategic approach.

Step 5: Use the direct derivative approach to find where $|T'(t)| > 3$

Computing $T'(t)$ directly:

$$T'(t) = 15 \cdot \cos\left(\frac{\pi t}{12}\right) \cdot \frac{\pi}{12} = \frac{15\pi}{12} \cos\left(\frac{\pi t}{12}\right) = \frac{5\pi}{4} \cos\left(\frac{\pi t}{12}\right)$$

The **maximum possible rate** is:

$$\max |T'(t)| = \frac{5\pi}{4} \approx \frac{5 \times 3.1416}{4} \approx \frac{15.708}{4} \approx 3.927°C/hour$$

This maximum occurs when $\cos\left(\frac{\pi t}{12}\right) = \pm 1$, i.e., at $t = 0$ and $t = 12$.

Step 6: Apply the MVT to guarantee activation

Now consider the interval $[0, 6]$ where the temperature rises from 20°C to 35°C.

$$\frac{T(6) - T(0)}{6 - 0} = 2.5°\text{C/hour}$$

The cooling system condition is $|T'(t)| > 3$. Although the MVT gives us the average rate, we can use a **tighter interval** to guarantee the MVT value exceeds 3.

Consider the interval $[0, 2]$:

$$T(2) = 20 + 15\sin\left(\frac{2\pi}{12}\right) = 20 + 15\sin\left(\frac{\pi}{6}\right) = 20 + 15 \times 0.5 = 27.5°\text{C}$$

$$\frac{T(2) - T(0)}{2 - 0} = \frac{27.5 - 20}{2} = \frac{7.5}{2} = 3.75°\text{C/hour}$$

By the MVT, since $T(t)$ is: - Continuous on $[0, 2]$ Yes…. (sum of continuous functions) - Differentiable on $(0, 2)$ Yes…. (sinusoidal function)

There **must exist** at least one $c \in (0, 2)$ such that:

$$T'(c) = 3.75°\text{C/hour} > 3°\text{C/hour}$$

Conclusion:

$$\boxed{T'(c) = 3.75 \text{ °C/hour} > 3 \text{ °C/hour for some } c \in (0,\ 2)}$$

The MVT **guarantees** the cooling system must activate during the interval $(0, 2)$ hours. Specifically, since the average rate of temperature rise over the first 2 hours is 3.75°C/hour — which exceeds the 3°C/hour threshold — the instantaneous rate must exceed 3°C/hour at some point in that window.

The cooling system is also guaranteed to activate near $t = 0$ and $t = 12$ where the instantaneous rate reaches its peak of $\approx 3.93°\text{C/hour}$.