Chapter 2: Integral Calculus

Advanced Topics

Dr. D Bhanu Prakash

Unit 2: Advanced Topics

Author: Dr. D Bhanu Prakash

Vignan's Foundation for Science, Technology and Research
Vadlamudi - 522 213, Guntur, Andhra Pradesh, India.

Course: Calculus and Ordinary Differential Equations
Chapter 2: Integral Calculus
Course Code: 25MT105

Part I: Integrals of the Form $\int_{x^2}^{x^3} f(t)\,dt$

When the limits of integration are functions of x, we apply the **Second Fundamental Theorem of Calculus** (Leibniz Rule):

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Example 1: $\displaystyle\int_{x^2}^{x^3} t\,dt$

Evaluate the integral and differentiate with respect to x.

Step 1: Compute the definite integral.

$$\int_{x^2}^{x^3} t\,dt = \left[\frac{t^2}{2}\right]_{x^2}^{x^3} = \frac{(x^3)^2}{2} - \frac{(x^2)^2}{2} = \frac{x^6}{2} - \frac{x^4}{2}$$

$$\boxed{\int_{x^2}^{x^3} t\,dt = \frac{x^6 - x^4}{2}}$$

Step 2: Differentiate using Leibniz Rule (verification).

Let $g(x) = x^2$, $h(x) = x^3$, $f(t) = t$.

$$\frac{d}{dx}\int_{x^2}^{x^3} t\,dt = (x^3)(3x^2) - (x^2)(2x) = 3x^5 - 2x^3$$

Verification: Differentiating the closed form:

$$\frac{d}{dx}\left(\frac{x^6 - x^4}{2}\right) = \frac{6x^5 - 4x^3}{2} = 3x^5 - 2x^3 \checkmark$$

Example 2: $\int_{x^2}^{x^3} t^2\,dt$

Evaluate the integral and find its derivative.

Step 1: Compute the definite integral.

$$\int_{x^2}^{x^3} t^2\,dt = \left[\frac{t^3}{3}\right]_{x^2}^{x^3} = \frac{(x^3)^3}{3} - \frac{(x^2)^3}{3} = \frac{x^9}{3} - \frac{x^6}{3}$$

$$\boxed{\int_{x^2}^{x^3} t^2\,dt = \frac{x^9 - x^6}{3}}$$

Step 2: Differentiate using Leibniz Rule (verification).

$$\frac{d}{dx}\int_{x^2}^{x^3} t^2\,dt = (x^3)^2 \cdot (3x^2) - (x^2)^2 \cdot (2x) = 3x^8 - 2x^5$$

Verification:

$$\frac{d}{dx}\left(\frac{x^9 - x^6}{3}\right) = \frac{9x^8 - 6x^5}{3} = 3x^8 - 2x^5 \checkmark$$

Example 3: $\int_{x^2}^{x^3} e^t\,dt$

Evaluate the integral and differentiate.

Step 1: Compute the definite integral.

$$\int_{x^2}^{x^3} e^t\,dt = [e^t]_{x^2}^{x^3} = e^{x^3} - e^{x^2}$$

$$\boxed{\int_{x^2}^{x^3} e^t\,dt = e^{x^3} - e^{x^2}}$$

Step 2: Differentiate using Leibniz Rule.

$$\frac{d}{dx}\int_{x^2}^{x^3} e^t\,dt = e^{x^3}\cdot(3x^2) - e^{x^2}\cdot(2x)$$

$$\boxed{\frac{d}{dx}\left(e^{x^3} - e^{x^2}\right) = 3x^2 e^{x^3} - 2x\,e^{x^2}}$$

Verification by direct differentiation:

$$\frac{d}{dx}\left(e^{x^3} - e^{x^2}\right) = e^{x^3}\cdot 3x^2 - e^{x^2}\cdot 2x = 3x^2 e^{x^3} - 2x\,e^{x^2} \checkmark$$

Part II: Reduction Formula for $\int \sin^n x\,dx$

Derivation

Let:

$$I_n = \int \sin^n x\,dx$$

Step 1: Split $\sin^n x$**.**

Write $\sin^n x = \sin^{n-1} x \cdot \sin x$ and apply **integration by parts**:

$$u = \sin^{n-1} x \qquad dv = \sin x\, dx$$
$$du = (n-1)\sin^{n-2} x \cos x\, dx \qquad v = -\cos x$$

Step 2: Apply integration by parts formula $\int u\, dv = uv - \int v\, du$**.**

$$I_n = -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\, dx$$

Step 3: Replace $\cos^2 x$ **using the Pythagorean identity** $\cos^2 x = 1 - \sin^2 x$:

$$I_n = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\,(1 - \sin^2 x)\, dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2} - (n-1)\, I_n$$

Step 4: Solve for I_n**.**

$$I_n + (n-1)\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$$

$$n\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$$

$$\boxed{I_n = \int \sin^n x\, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx}$$

Base Cases

$$I_0 = \int 1\, dx = x + C$$

$$I_1 = \int \sin x \, dx = -\cos x + C$$

Definite Integral Form (Wallis' Formula)

For the definite integral over $[0, \pi/2]$:

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \times \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Part III: Reduction Formula for $\int \cos^n x \, dx$

Derivation

Let:

$$J_n = \int \cos^n x \, dx$$

Step 1: Split $\cos^n x$.

Write $\cos^n x = \cos^{n-1} x \cdot \cos x$ and apply **integration by parts**:

$$u = \cos^{n-1} x \qquad dv = \cos x \, dx$$
$$du = -(n-1) \cos^{n-2} x \sin x \, dx \qquad v = \sin x$$

Step 2: Apply integration by parts formula.

$$J_n = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \, dx$$

Step 3: Replace $\sin^2 x$ using $\sin^2 x = 1 - \cos^2 x$:

$$J_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$J_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$J_n = \cos^{n-1} x \sin x + (n-1)\, J_{n-2} - (n-1)\, J_n$$

Step 4: Solve for J_n.

$$J_n + (n-1)\, J_n = \cos^{n-1} x \sin x + (n-1)\, J_{n-2}$$

$$n\, J_n = \cos^{n-1} x \sin x + (n-1)\, J_{n-2}$$

$$\boxed{J_n = \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx}$$

Base Cases

$$J_0 = \int 1 \, dx = x + C$$

$$J_1 = \int \cos x \, dx = \sin x + C$$

Definite Integral Form (Wallis' Formula)

For the definite integral over $[0, \pi/2]$, by symmetry with the sine formula:

$$\int_0^{\pi/2} \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \times \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Summary Table

Formula	Result
$\int_{x^2}^{x^3} t \, dt$	$\frac{x^6 - x^4}{2}$
$\int_{x^2}^{x^3} t^2 \, dt$	$\frac{x^9 - x^6}{3}$

Formula	Result
$\int_{x^2}^{x^3} e^t \, dt$	$e^{x^3} - e^{x^2}$
$\int \sin^n x \, dx$	$-\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$
$\int \cos^n x \, dx$	$\frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} J_{n-2}$

> **Note:** Both the $\sin^n$ and $\cos^n$ reduction formulas share the same structural form — they reduce the power by 2 at each step, eventually terminating at I_0 or I_1.