Chapter 2: Integral Calculus

Lecture

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Unit 2: Lecture Notes

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Topic 7: Integration as the Limit of a Sum

Introduction

Integration is fundamentally the process of finding the area under a curve by dividing it into infinitely many thin strips and summing their areas.

7.1 The Basic Idea

Consider a function $f(x)$ defined on $[a, b]$. We want to find the area under the curve between $x = a$ and $x = b$.

Step 1: Divide $[a, b]$ into n equal subintervals, each of width:

$$\Delta x = \frac{b-a}{n}$$

Step 2: The subinterval endpoints are:

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_n = b$$

Step 3: On each subinterval, approximate the area by a rectangle of height $f(x_i)$ and width Δx.

Step 4: Sum all n rectangles:

$$S_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

Step 5: Take the limit as $n \to \infty$ (rectangles become infinitely thin):

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

7.2 Useful Summation Formulas

The following standard results are used when evaluating these limits:

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Detailed Example

Problem: Evaluate $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Solution:

Step 1: Set $a = 0, b = 2$, so:

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

Step 2: The i-th right-endpoint is:

$$x_i = a + i\Delta x = 0 + i \cdot \frac{2}{n} = \frac{2i}{n}$$

Step 3: Write the Riemann sum

$$S_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 1\right] \cdot \frac{2}{n}$$

$$= \sum_{i=1}^{n} \left[\frac{4i^2}{n^2} + 1\right] \cdot \frac{2}{n}$$

$$= \sum_{i=1}^{n} \left[\frac{8i^2}{n^3} + \frac{2}{n}\right]$$

$$= \frac{8}{n^3} \sum_{i=1}^{n} i^2 + \frac{2}{n} \sum_{i=1}^{n} 1$$

Step 4: Substitute summation formulas

$$= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{2}{n} \cdot n$$

$$= \frac{8n(n+1)(2n+1)}{6n^3} + 2$$

$$= \frac{4(n+1)(2n+1)}{3n^2} + 2$$

Step 5: Take the limit as $n \to \infty$

$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty} \left[\frac{4(n+1)(2n+1)}{3n^2} + 2\right]$$

Expand the numerator: $(n+1)(2n+1) = 2n^2 + 3n + 1$

$$= \lim_{n\to\infty} \left[\frac{4(2n^2+3n+1)}{3n^2} + 2\right]$$

$$= \lim_{n\to\infty} \left[\frac{8n^2+12n+4}{3n^2} + 2\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{3} + \frac{12}{3n} + \frac{4}{3n^2} + 2\right]$$

$$= \frac{8}{3} + 0 + 0 + 2 = \frac{8}{3} + \frac{6}{3} = \frac{14}{3}$$

Answer: $\int_0^2 (x^2+1)\,dx = \frac{14}{3}$

Topic 8: Fundamental Theorem of Integral Calculus (FTIC)

Introduction

The **Fundamental Theorem of Integral Calculus** is one of the most important results in all of mathematics. It establishes the powerful connection between differentiation and integration, allowing us to avoid tedious limit calculations.

8.1 Part 1 of FTIC

Statement: If $f(x)$ is continuous on $[a, b]$ and we define:

$$F(x) = \int_a^x f(t)\,dt$$

then $F(x)$ is differentiable on (a, b) and:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Meaning: Differentiation and integration are inverse operations.

8.2 Part 2 of FTIC (Evaluation Theorem)

Statement: If $f(x)$ is continuous on $[a, b]$ and $G(x)$ is any antiderivative of $f(x)$ (i.e., $G'(x) = f(x)$), then:

$$\int_a^b f(x)\,dx = G(b) - G(a) = \Big[G(x)\Big]_a^b$$

This is the theorem that makes calculus computable.

8.3 Extended Form (Chain Rule Version)

If the limits involve functions of x:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$

Detailed Example

Problem: Evaluate $\int_1^3 (x^3 - 4x + 2)\,dx$ using FTIC.

Solution:

Step 1: Find the antiderivative

$$G(x) = \int (x^3 - 4x + 2)\,dx = \frac{x^4}{4} - 2x^2 + 2x + C$$

Step 2: Apply FTIC

$$\int_1^3 (x^3 - 4x + 2)\,dx = \left[\frac{x^4}{4} - 2x^2 + 2x\right]_1^3$$

Step 3: Evaluate at upper limit $x = 3$

$$G(3) = \frac{81}{4} - 2(9) + 2(3) = \frac{81}{4} - 18 + 6 = \frac{81}{4} - 12 = \frac{81 - 48}{4} = \frac{33}{4}$$

Step 4: Evaluate at lower limit $x = 1$

$$G(1) = \frac{1}{4} - 2(1) + 2(1) = \frac{1}{4} - 2 + 2 = \frac{1}{4}$$

Step 5: Subtract

$$\int_1^3 (x^3 - 4x + 2)\,dx = G(3) - G(1) = \frac{33}{4} - \frac{1}{4} = \frac{32}{4} = 8$$

Answer: 8

Topic 9: Mean Value Theorems for Integrals

Introduction

Just as the differential calculus has mean value theorems, integral calculus has its own mean value theorems. These are extremely important in mathematical analysis and help bridge the gap between exact and average values of functions.

9.1 First Mean Value Theorem for Integrals

Statement: If $f(x)$ is continuous on $[a, b]$, then there exists at least one point $c \in (a, b)$ such that:

$$\int_a^b f(x)\,dx = f(c) \cdot (b - a)$$

Equivalently:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

This value $f(c)$ is called the **average value** (or mean value) of f on $[a, b]$.

Geometric Meaning: There is a rectangle with base $(b - a)$ and height $f(c)$ whose area equals the area under the curve.

9.2 Second Mean Value Theorem for Integrals (Generalized)

Statement: If $f(x)$ is continuous and $g(x)$ is non-negative and integrable on $[a, b]$, then there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx$$

Problem 9.1

Problem: Verify the First Mean Value Theorem for $f(x) = x^2 + x + 1$ on $[1, 4]$ and find the value of c.

Solution:

Step 1: Confirm continuity

$f(x) = x^2 + x + 1$ is a polynomial — continuous everywhere.

Step 2: Compute the definite integral

$$\int_1^4 (x^2 + x + 1)\,dx = \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_1^4$$

At $x = 4$:

$$\frac{64}{3} + \frac{16}{2} + 4 = \frac{64}{3} + 8 + 4 = \frac{64}{3} + 12 = \frac{64+36}{3} = \frac{100}{3}$$

At $x = 1$:

$$\frac{1}{3} + \frac{1}{2} + 1 = \frac{2}{6} + \frac{3}{6} + \frac{6}{6} = \frac{11}{6}$$

$$\int_1^4 (x^2 + x + 1)\,dx = \frac{100}{3} - \frac{11}{6} = \frac{200}{6} - \frac{11}{6} = \frac{189}{6} = \frac{63}{2}$$

Step 3: Apply the Mean Value Theorem

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\cdot\frac{63}{2} = \frac{63}{6} = \frac{21}{2}$$

Step 4: Solve for c

$$f(c) = c^2 + c + 1 = \frac{21}{2}$$

$$c^2 + c + 1 - \frac{21}{2} = 0$$

$$c^2 + c - \frac{19}{2} = 0$$

$$2c^2 + 2c - 19 = 0$$

Using the quadratic formula:

$$c = \frac{-2 \pm \sqrt{4+152}}{4} = \frac{-2 \pm \sqrt{156}}{4} = \frac{-2 \pm 2\sqrt{39}}{4} = \frac{-1 \pm \sqrt{39}}{2}$$

Step 5: Check which value lies in $(1, 4)$

$\sqrt{39} \approx 6.245$

$c = \frac{-1+6.245}{2} \approx \frac{5.245}{2} \approx 2.62$ (lies in $(1, 4)$)

$c = \frac{-1-6.245}{2} \approx -3.62$ (not in $(1, 4)$)

Answer: $c = \dfrac{-1+\sqrt{39}}{2} \approx 2.62$

Problem 9.2

Problem: Find the average value of $f(x) = \sin x$ on $[0, \pi]$ and find the point c guaranteed by the Mean Value Theorem.

Solution:

Step 1: Compute the integral

$$\int_0^\pi \sin x \, dx = \Big[-\cos x \Big]_0^\pi = -\cos \pi + \cos 0 = -(-1) + 1 = 2$$

Step 2: Find the average value

$$f_{avg} = \frac{1}{\pi - 0} \cdot 2 = \frac{2}{\pi}$$

Step 3: Solve for c

$$\sin c = \frac{2}{\pi}$$

$$c = \arcsin\left(\frac{2}{\pi}\right) \approx \arcsin(0.6366) \approx 0.6901 \text{ radians}$$

Step 4: Verify $c \in (0, \pi)$

Since $0 < 0.6901 < \pi$, the point is valid.

But also note: $\sin x = \frac{2}{\pi}$ in $(0, \pi)$ gives two solutions: - $c_1 \approx 0.6901$ - $c_2 = \pi - 0.6901 \approx 2.451$

Both are valid points in $(0, \pi)$.

Answer: Average value $= \dfrac{2}{\pi} \approx 0.637$; $c \approx 0.690$ or $c \approx 2.451$

Problem 9.3

Problem: Verify the First Mean Value Theorem for $f(x) = e^x$ on $[0, 1]$ and find c exactly.

Solution:

Step 1: Confirm continuity

$f(x) = e^x$ is continuous everywhere.

Step 2: Compute the integral

$$\int_0^1 e^x \, dx = \Big[e^x\Big]_0^1 = e^1 - e^0 = e - 1$$

Step 3: Find average value

$$f(c) = \frac{1}{1-0}(e-1) = e-1$$

Step 4: Solve for c

$$e^c = e-1$$

$$c = \ln(e-1)$$

Step 5: Numerical check

$e \approx 2.718$, so $e-1 \approx 1.718$

$c = \ln(1.718) \approx 0.541$

Verify: $0 < 0.541 < 1$

Answer: $c = \ln(e-1) \approx 0.541$

Problem 9.4

Problem: Find the value of c guaranteed by the Mean Value Theorem for $f(x) = \frac{1}{x}$ on $[1, e]$.

Solution:

Step 1: Continuity

$f(x) = \frac{1}{x}$ is continuous on $[1, e]$ since $x > 0$ throughout.

Step 2: Compute the integral

$$\int_1^e \frac{1}{x}\,dx = \Big[\ln x\Big]_1^e = \ln e - \ln 1 = 1 - 0 = 1$$

Step 3: Average value

$$f(c) = \frac{1}{e-1} \cdot 1 = \frac{1}{e-1}$$

Step 4: Solve for c

$$\frac{1}{c} = \frac{1}{e-1}$$

$$c = e - 1 \approx 1.718$$

Step 5: Verify $c \in (1, e)$

Since $1 < 1.718 < 2.718$, i.e., $c \in (1, e)$.

Answer: $c = e - 1 \approx 1.718$

Problem 9.5

Problem: Apply the Second Mean Value Theorem to $\int_0^{\pi/2} x \cos x \, dx$ where $f(x) = \cos x$ and $g(x) = x$.

Solution:

Step 1: Compute the integral directly to find the LHS

Using integration by parts:

$$\int_0^{\pi/2} x \cos x \, dx$$

Let $u = x$, $dv = \cos x \, dx$ so $du = dx$, $v = \sin x$:

$$= \Big[x \sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx$$

$$= \left(\frac{\pi}{2} \cdot \sin\frac{\pi}{2} - 0\right) - \Big[-\cos x\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2}(1) + \Big[\cos x\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{\pi}{2} + (0 - 1) = \frac{\pi}{2} - 1$$

Step 2: Apply the Second Mean Value Theorem

$$\int_0^{\pi/2} f(x)g(x)\,dx = f(c)\int_0^{\pi/2} g(x)\,dx$$

$$\int_0^{\pi/2} x\,dx = \left[\frac{x^2}{2}\right]_0^{\pi/2} = \frac{\pi^2}{8}$$

So the theorem gives:

$$\frac{\pi}{2} - 1 = \cos(c)\cdot\frac{\pi^2}{8}$$

$$\cos(c) = \frac{8\left(\frac{\pi}{2}-1\right)}{\pi^2} = \frac{8(\pi-2)}{2\pi^2} = \frac{4(\pi-2)}{\pi^2}$$

$$\cos(c) = \frac{4\pi-8}{\pi^2} \approx \frac{12.566-8}{9.870} \approx \frac{4.566}{9.870} \approx 0.4626$$

$$c = \text{arccos}(0.4626) \approx 1.088 \text{ radians}$$

Verify: $0 < 1.088 < \pi/2 \approx 1.571$

Answer: $c \approx 1.088$ radians

Topic 10: Reduction Formulae

Introduction

A **reduction formula** expresses an integral involving a power n in terms of the same integral with a lower power. This allows us to break down complex integrals step by step until we reach a simple base case.

General Form:

$$I_n = (\text{some expression in } n)\times I_{n-2} \quad \text{or} \quad I_{n-1}$$

10.1 Reduction Formula for $\int \sin^n x\,dx$

Formula:

$$I_n = \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

For definite integrals on $\left[0, \frac{\pi}{2}\right]$ (Wallis' Formula):

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \times \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Problem: Using the reduction formula, evaluate $\int_0^{\pi/2} \sin^5 x \, dx$.

Solution:

Using Wallis' formula with $n = 5$ (odd):

$$I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{8}{15}$$

Verification step-by-step:

$$I_5 = \frac{4}{5} I_3$$

$$I_3 = \frac{2}{3} I_1$$

$$I_1 = \int_0^{\pi/2} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi/2} = 0 + 1 = 1$$

So: $I_3 = \frac{2}{3} \times 1 = \frac{2}{3}$

And: $I_5 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$

Answer: $\frac{8}{15}$

10.2 Reduction Formula for $\int \cos^n x \, dx$

Formula:

$$J_n = \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} J_{n-2}$$

Wallis' Formula (same as $\sin$):

$$J_n = \int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \times \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Problem: Evaluate $\int_0^{\pi/2} \cos^6 x \, dx$.

Solution:

Using Wallis' formula with $n = 6$ (even):

$$I_6 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Step by step:

$I_6 = \frac{5}{6} I_4$

$I_4 = \frac{3}{4} I_2$

$I_2 = \frac{1}{2} I_0$

$I_0 = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$

Therefore:

$$I_2 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

$$I_4 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}$$

$$I_6 = \frac{5}{6} \cdot \frac{3\pi}{16} = \frac{15\pi}{96} = \frac{5\pi}{32}$$

Answer: $\frac{5\pi}{32}$

10.3 Reduction Formula for $\int x^n e^x \, dx$

Formula:

$$I_n = \int x^n e^x \, dx = x^n e^x - nI_{n-1}$$

Problem: Using the reduction formula, evaluate $\int x^4 e^x \, dx$.

Solution:

Applying the formula repeatedly:

$$I_4 = x^4 e^x - 4I_3$$

$$I_3 = x^3 e^x - 3I_2$$

$$I_2 = x^2 e^x - 2I_1$$

$$I_1 = xe^x - I_0$$

$$I_0 = \int e^x \, dx = e^x$$

Building back up:

$$I_1 = xe^x - e^x = e^x(x - 1)$$

$$I_2 = x^2 e^x - 2e^x(x - 1) = e^x(x^2 - 2x + 2)$$

$$I_3 = x^3 e^x - 3e^x(x^2 - 2x + 2) = e^x(x^3 - 3x^2 + 6x - 6)$$

$$I_4 = x^4e^x - 4e^x(x^3 - 3x^2 + 6x - 6) = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)$$

Answer: $\int x^4e^x\,dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C$

10.4 Reduction Formula for $\int \sin^m x \cos^n x\,dx$

Formula:

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \times K$$

where $K = \frac{\pi}{2}$ if both m and n are even, and $K = 1$ otherwise.
This is the **Beta function formula** (Wallis' generalization).

Problem 10.4: Mixed $\sin^m \cos^n$ Reduction

Problem: Evaluate $\int_0^{\pi/2} \sin^3 x \cos^4 x\,dx$.

Solution:

Here $m = 3$ (odd), $n = 4$ (even). Since m is odd, $K = 1$.

$$I_{3,4} = \frac{(m-1)(n-1)(n-3)}{(m+n)(m+n-2)(m+n-4)} \times 1$$

$$= \frac{(2)(3)(1)}{(7)(5)(3)} = \frac{6}{105} = \frac{2}{35}$$

Answer: $\frac{2}{35}$

Problem: Derive the reduction formula for $I_n = \int x^n \sin x\,dx$ and use it to find $\int x^3 \sin x\,dx$.

Solution:

Deriving the formula using integration by parts:

Let $u = x^n$, $dv = \sin x\,dx$, so $du = nx^{n-1}\,dx$, $v = -\cos x$:

$$I_n = -x^n \cos x + n \int x^{n-1} \cos x\,dx$$

Now apply IBP again to $\int x^{n-1} \cos x \, dx$:

Let $u = x^{n-1}$, $dv = \cos x \, dx$, so $du = (n-1)x^{n-2} \, dx$, $v = \sin x$:

$$\int x^{n-1} \cos x \, dx = x^{n-1} \sin x - (n-1) \int x^{n-2} \sin x \, dx$$

Therefore the **reduction formula** is:

$$\boxed{I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}}$$

Applying for $n = 3$:

$$I_3 = -x^3 \cos x + 3x^2 \sin x - 6I_1$$

$$I_1 = -x \cos x + \sin x$$

$$I_3 = -x^3 \cos x + 3x^2 \sin x - 6(-x \cos x + \sin x) + C$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Answer: $\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

Topic 11: Improper Integrals

Introduction

A **definite integral** $\int_a^b f(x) \, dx$ is called **improper** if: 1. One or both limits of integration are $\pm\infty$, or 2. The integrand $f(x)$ is unbounded (has a vertical asymptote) at some point in $[a, b]$.

11.1 Type I: Infinite Limits of Integration

Case 1: Upper limit infinite:

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Case 2: Lower limit infinite:

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

Case 3: Both limits infinite:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

for any convenient c (usually $c = 0$).

The integral **converges** if the limit is finite; it **diverges** if the limit is infinite or does not exist.

Problem 11.1: Type I — Upper Limit Infinite

Problem: Evaluate $\int_1^{\infty} \frac{1}{x^3}\,dx$.

Solution:

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t x^{-3}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-2}}{-2}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}$$

Answer: The integral **converges** to $\frac{1}{2}$.

Problem 11.2: Type I — Both Limits Infinite

Problem: Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

Solution:

Split at $c = 0$:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

Left part:

$$\lim_{s\to-\infty} \Big[\arctan x\Big]_s^0 = \arctan 0 - \lim_{s\to-\infty} \arctan s = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Right part:

$$\lim_{t\to\infty} \Big[\arctan x\Big]_0^t = \lim_{t\to\infty} \arctan t - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

Total:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Answer: The integral **converges** to π.

11.2 Type II: Discontinuous Integrand

Case 1: Discontinuity at the right endpoint b:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

Case 2: Discontinuity at the left endpoint a:

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

Case 3: Discontinuity at an interior point $c \in (a, b)$:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Problem 11.3: Type II — Discontinuity at Endpoint

Problem: Evaluate $\int_0^1 \dfrac{1}{\sqrt{1-x}}\,dx$.

Solution:

The integrand has a discontinuity at $x = 1$ (right endpoint).

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{t\to 1^-} \int_0^t (1-x)^{-1/2}\,dx$$

$$= \lim_{t\to 1^-} \left[\frac{(1-x)^{1/2}}{-\frac{1}{2}\cdot(-1)}\right]_0^t = \lim_{t\to 1^-} \left[-2\sqrt{1-x}\right]_0^t$$

$$= \lim_{t\to 1^-} \left(-2\sqrt{1-t} + 2\sqrt{1-0}\right) = -2(0) + 2(1) = 2$$

Answer: The integral **converges** to 2.

Problem 11.4: Type II — Discontinuity at Interior Point

Problem: Evaluate $\int_{-1}^1 \dfrac{1}{x^{2/3}}\,dx$.

Solution:

The integrand is discontinuous at $x = 0$ (interior point).

Split at $c = 0$:

$$\int_{-1}^1 x^{-2/3}\,dx = \int_{-1}^0 x^{-2/3}\,dx + \int_0^1 x^{-2/3}\,dx$$

Right part:

$$\lim_{t\to 0^+} \int_t^1 x^{-2/3}\,dx = \lim_{t\to 0^+} \left[\frac{x^{1/3}}{1/3}\right]_t^1 = \lim_{t\to 0^+} \left[3x^{1/3}\right]_t^1 = 3(1) - 3(0) = 3$$

Left part:

$$\lim_{s \to 0^-} \int_{-1}^{s} x^{-2/3}\, dx = \lim_{s \to 0^-} \left[3x^{1/3}\right]_{-1}^{s} = 3(0) - 3(-1) = 3$$

Total:

$$\int_{-1}^{1} x^{-2/3}\, dx = 3 + 3 = 6$$

Answer: The integral **converges** to 6.

Problem 11.5: Divergent Improper Integral

Problem: Show that $\int_1^\infty \dfrac{1}{\sqrt{x}}\, dx$ diverges.

Solution:

$$\int_1^\infty \frac{1}{\sqrt{x}}\, dx = \lim_{t \to \infty} \int_1^t x^{-1/2}\, dx = \lim_{t \to \infty} \left[2\sqrt{x}\right]_1^t = \lim_{t \to \infty} (2\sqrt{t} - 2)$$

As $t \to \infty$, $2\sqrt{t} \to \infty$.

Answer: The integral **diverges** to $+\infty$.

Topic 12: Tests of Convergence

Introduction

When we cannot evaluate an improper integral directly, we use **tests of convergence** to determine whether it converges or diverges. These are particularly useful for complicated integrands.

12.1 Summary Table of All Convergence Tests

Test	Statement	When to Use	Key Condition
p-Test (Type I)	$\int_1^\infty \frac{dx}{x^p}$ converges if $p > 1$; diverges if $p \le 1$	Integrands of power type on $[1, \infty)$	Compare with $\frac{1}{x^p}$

Test	Statement	When to Use	Key Condition
p-Test (Type II)	$\int_0^1 \frac{dx}{x^p}$ converges if $p < 1$; diverges if $p \geq 1$	Integrands with singularity at $x = 0$	Compare with $\frac{1}{x^p}$ near 0
Comparison Test	If $0 \leq f(x) \leq g(x)$: (i) $\int g$ converges $\Rightarrow \int f$ converges; (ii) $\int f$ diverges $\Rightarrow \int g$ diverges	f is bounded by known function	Need $f \leq g$ explicitly
Limit Comparison Test	If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ where $0 < L < \infty$, then $\int f$ and $\int g$ behave the same	f and g are asymptotically similar	L must be finite and nonzero
Absolute Convergence Test	If $\int_a^\infty \|f(x)\| \, dx$ converges, then $\int_a^\infty f(x) \, dx$ converges	Integrals with sign changes	Works via Comparison Test
Dirichlet's Test	If $f(x) \to 0$ monotonically and $\left\|\int_a^t g\right\| \leq M$, then $\int_a^\infty fg \, dx$ converges	Oscillating integrands like $\frac{\sin x}{x}$	Requires bounded antiderivative

12.2 Key Differences Between the Tests

Feature	Comparison Test	Limit Comparison Test
Requires explicit inequality	Yes: $f(x) \leq g(x)$	No
Works with asymptotically equivalent functions	Only if inequality holds	Yes — ideal for this
Easier to apply when…	Inequality is obvious	Direct comparison is hard

Feature	Comparison Test	Limit Comparison Test
Both functions must be non-negative	Yes	Yes
If limit $= 0$	Not applicable	$\int g$ converges $\Rightarrow \int f$ converges
If limit $= \infty$	Not applicable	$\int g$ diverges $\Rightarrow \int f$ diverges

12.3 The p-Test (Benchmark Integrals)

These are the most important reference integrals:

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{converges if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$$

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} \dfrac{1}{1-p} & \text{converges if } p < 1 \\ \text{diverges} & \text{if } p \ge 1 \end{cases}$$

Note: Think of it as a simple rule — for $\int_1^\infty$, the exponent must be **greater than 1** to converge; for $\int_0^1$, the exponent must be **less than 1** to converge.

Problem 12.1: p-Test

Problem: Determine the convergence of $\int_2^\infty \frac{1}{x^{3/2}}\, dx$.

Solution:

This is a p-integral with $p = \frac{3}{2}$.

Since $p = \frac{3}{2} > 1$, by the p-Test, the integral **converges**.

Verification (computing directly):

$$\int_2^\infty x^{-3/2}\, dx = \lim_{t\to\infty} \left[\frac{x^{-1/2}}{-1/2}\right]_2^t = \lim_{t\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_2^t = \lim_{t\to\infty} \left(-\frac{2}{\sqrt{t}} + \frac{2}{\sqrt{2}}\right) = 0 + \sqrt{2} = \sqrt{2}$$

Answer: **Converges** to $\sqrt{2}$.

Problem 12.2: Direct Comparison Test

Problem: Test the convergence of $\int_1^\infty \frac{1}{x^2 + \sin^2 x}\,dx$.

Solution:

Step 1: Find a comparison function

Observe that $\sin^2 x \geq 0$ for all x, so:

$$x^2 + \sin^2 x \geq x^2$$

Therefore:

$$\frac{1}{x^2 + \sin^2 x} \leq \frac{1}{x^2}$$

Step 2: Check the comparison integral

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{is a p-integral with } p = 2 > 1 \Rightarrow \text{converges}$$

Step 3: Apply Comparison Test

Since $0 \leq \frac{1}{x^2 + \sin^2 x} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges,

by the Comparison Test, $\int_1^\infty \frac{1}{x^2 + \sin^2 x}\,dx$ also **converges**.

Problem 12.3: Limit Comparison Test

Problem: Determine the convergence of $\int_1^\infty \frac{x^2 + 3}{x^4 - 2x + 5}\,dx$.

Solution:

Step 1: Identify the dominant behavior

For large x:

$$\frac{x^2 + 3}{x^4 - 2x + 5} \approx \frac{x^2}{x^4} = \frac{1}{x^2}$$

So choose comparison function $g(x) = \frac{1}{x^2}$.

Step 2: Compute the limit

$$L = \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{x^2 + 3}{x^4 - 2x + 5} \cdot \frac{x^2}{1}$$

$$= \lim_{x \to \infty} \frac{x^4 + 3x^2}{x^4 - 2x + 5}$$

Divide numerator and denominator by x^4:

$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x^2}}{1 - \frac{2}{x^3} + \frac{5}{x^4}} = \frac{1 + 0}{1 - 0 + 0} = 1$$

Step 3: Apply the test

Since $0 < L = 1 < \infty$ and $\int_1^\infty \frac{1}{x^2}\, dx$ converges (p-test, $p = 2 > 1$),

by the Limit Comparison Test, $\int_1^\infty \frac{x^2 + 3}{x^4 - 2x + 5}\, dx$ also **converges**.

Problem 12.4: Comparison Test (Divergence)

Problem: Show that $\int_1^\infty \frac{1}{\sqrt{x^2 + 1}}\, dx$ diverges.

Solution:

Step 1: Find a lower bound

Observe: $x^2 + 1 \leq x^2 + x^2 = 2x^2$ for $x \geq 1$

So: $\sqrt{x^2 + 1} \leq \sqrt{2x^2} = \sqrt{2}\, x$

Therefore:

$$\frac{1}{\sqrt{x^2 + 1}} \geq \frac{1}{\sqrt{2}\, x}$$

Step 2: Check the comparison integral

$$\int_1^\infty \frac{1}{\sqrt{2}\, x}\, dx = \frac{1}{\sqrt{2}} \int_1^\infty \frac{dx}{x}$$

This is a p-integral with $p = 1$. By p-Test, it **diverges**.

Step 3: Apply Comparison Test

Since $\dfrac{1}{\sqrt{x^2+1}} \geq \dfrac{1}{\sqrt{2}\,x}$ and $\int_1^\infty \dfrac{1}{\sqrt{2}\,x}\,dx$ diverges,

by the Comparison Test, $\int_1^\infty \dfrac{1}{\sqrt{x^2+1}}\,dx$ also **diverges**.

Problem 12.5: Type II Convergence (Near a Singularity)

Problem: Test the convergence of $\int_0^1 \dfrac{1}{x^{2/3}(1+x)}\,dx$.

Solution:

The integrand has a singularity at $x = 0$. Use the p-Test for Type II integrals.

Step 1: Behavior near $x = 0$

As $x \to 0^+$: $(1+x) \to 1$, so:

$$\frac{1}{x^{2/3}(1+x)} \approx \frac{1}{x^{2/3}}$$

Step 2: Apply Limit Comparison

$$L = \lim_{x\to 0^+} \frac{\frac{1}{x^{2/3}(1+x)}}{\frac{1}{x^{2/3}}} = \lim_{x\to 0^+} \frac{1}{1+x} = 1$$

Step 3: Check the reference integral

$\int_0^1 \dfrac{1}{x^{2/3}}\,dx$ is a Type II p-integral with $p = \dfrac{2}{3} < 1$. By p-Test, it **converges**.

Step 4: Conclude

Since $L = 1$ (finite and nonzero) and reference integral converges, by Limit Comparison Test, $\int_0^1 \dfrac{1}{x^{2/3}(1+x)}\,dx$ also **converges**.

Topic 13: Rectification (Arc Length)

Introduction

Rectification is the process of finding the **exact length of a curve**. The term comes from "rectify" — meaning to make straight. The arc length formula is derived by approximating the curve using many tiny straight line segments and taking the limit.

13.1 Derivation of Arc Length Formula

Consider a curve $y = f(x)$ from $x = a$ to $x = b$.

Step 1: Divide the interval $[a, b]$ into n small segments. Each segment has a chord (straight line) connecting two nearby points on the curve.

Step 2: A small element of arc length is:

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Step 3: Sum all elements:

$$L = \int_a^b ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

13.2 Arc Length Formulas in Different Forms

For $y = f(x)$:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For $x = g(y)$:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

For parametric curves $x = x(t), y = y(t), t \in [\alpha, \beta]$:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

For polar curves $r = f(\theta), \theta \in [\theta_1, \theta_2]$:

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Problem 13.1: Arc Length of $y = f(x)$

Problem: Find the arc length of the curve $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 4$.

Solution:

Step 1: Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}x^{1/2} = x^{1/2} = \sqrt{x}$$

Step 2: Set up the arc length integral

$$L = \int_0^4 \sqrt{1 + \left(\sqrt{x}\right)^2}\, dx = \int_0^4 \sqrt{1+x}\, dx$$

Step 3: Evaluate

$$L = \int_0^4 (1+x)^{1/2}\, dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^4 = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^4$$

$$= \frac{2}{3}(5)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3}\left(5\sqrt{5} - 1\right)$$

$$= \frac{2}{3}(11.180 - 1) = \frac{2}{3}(10.180) \approx 6.787$$

Answer: $L = \frac{2}{3}(5\sqrt{5} - 1) \approx 6.787$ units

Problem 13.2: Arc Length of a Parabola

Problem: Find the length of the parabola $y^2 = 4x$ from the vertex to the point $(4, 4)$.

Solution:

Express as $x = g(y)$:

$$y^2 = 4x \implies x = \frac{y^2}{4}$$

Step 1: Find $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{y}{2}$$

Step 2: Set limits: from the vertex $(0, 0)$ to $(4, 4)$, y goes from 0 to 4.

Step 3: Arc length formula

$$L = \int_0^4 \sqrt{1 + \left(\frac{y}{2}\right)^2}\, dy = \int_0^4 \sqrt{1 + \frac{y^2}{4}}\, dy = \int_0^4 \frac{1}{2}\sqrt{4 + y^2}\, dy$$

Step 4: Use the standard formula $\int \sqrt{a^2 + y^2}\, dy = \frac{y\sqrt{a^2 + y^2}}{2} + \frac{a^2}{2} \ln\left|y + \sqrt{a^2 + y^2}\right| + C$

With $a = 2$:

$$L = \frac{1}{2}\left[\frac{y\sqrt{4 + y^2}}{2} + \frac{4}{2}\ln\left|y + \sqrt{4 + y^2}\right|\right]_0^4$$

$$= \frac{1}{2}\left[\frac{y\sqrt{4 + y^2}}{2} + 2\ln\left(y + \sqrt{4 + y^2}\right)\right]_0^4$$

At $y = 4$: $\sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

$$= \frac{1}{2}\left[\frac{4 \cdot 2\sqrt{5}}{2} + 2\ln(4 + 2\sqrt{5}) - 0 - 2\ln(0 + 2)\right]$$

$$= \frac{1}{2}\left[4\sqrt{5} + 2\ln\left(\frac{4 + 2\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{2}\left[4\sqrt{5} + 2\ln(2 + \sqrt{5})\right]$$

$$= 2\sqrt{5} + \ln(2 + \sqrt{5})$$

$$\approx 2(2.236) + \ln(4.236) \approx 4.472 + 1.443 \approx 5.915$$

Answer: $L = 2\sqrt{5} + \ln(2 + \sqrt{5}) \approx 5.915$ units

Problem 13.3: Parametric Arc Length

Problem: Find the total length of the curve defined parametrically by $x = a\cos^3 t$, $y = a\sin^3 t$ (Astroid).

Solution:

Step 1: Differentiate

$$\frac{dx}{dt} = -3a\cos^2 t \sin t$$

$$\frac{dy}{dt} = 3a\sin^2 t \cos t$$

Step 2: Compute the integrand

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\cos^4 t \sin^2 t + 9a^2\sin^4 t \cos^2 t$$

$$= 9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t) = 9a^2\sin^2 t\cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3a|\sin t\cos t| = \frac{3a}{2}|\sin 2t|$$

Step 3: Integrate over one period $[0, 2\pi]$ (using symmetry: 4 times $[0, \pi/2]$)

$$L = 4\int_0^{\pi/2} \frac{3a}{2}\sin 2t\,dt = 6a\int_0^{\pi/2} \sin 2t\,dt$$

$$= 6a\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2} = 6a\left(-\frac{\cos\pi}{2} + \frac{\cos 0}{2}\right)$$

$$= 6a\left(\frac{1}{2} + \frac{1}{2}\right) = 6a$$

Answer: Total arc length of the astroid $= 6a$ units

Problem 13.4: Polar Arc Length

Problem: Find the arc length of the cardioid $r = a(1 + \cos\theta)$ for $0 \leq \theta \leq 2\pi$.

Solution:

Step 1: Compute $\dfrac{dr}{d\theta}$

$$\frac{dr}{d\theta} = -a\sin\theta$$

Step 2: Set up the arc length integral

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{(1 + 2\cos\theta + \cos^2\theta) + \sin^2\theta}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta$$

Step 3: Simplify using half-angle identity: $1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$

$$= a\int_0^{2\pi} \sqrt{4\cos^2\frac{\theta}{2}}\, d\theta = 2a\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta$$

Step 4: Split at $\theta = \pi$ (where $\cos\frac{\theta}{2}$ changes sign):

$$= 2a\left[\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta + \int_{\pi}^{2\pi} \left(-\cos\frac{\theta}{2}\right) d\theta\right]$$

$$= 2a \left[\left[2\sin\frac{\theta}{2} \right]_0^{\pi} + \left[-2\sin\frac{\theta}{2} \right]_{\pi}^{2\pi} \right]$$

$$= 2a \left[(2\sin\frac{\pi}{2} - 0) + (-2\sin\pi + 2\sin\frac{\pi}{2}) \right]$$

$$= 2a\left[2 + (0 + 2)\right] = 2a \times 4 = 8a$$

Answer: Total arc length of the cardioid $= 8a$ units

13.3 Summary of Rectification

Curve Type	Formula	Key Derivative
$y = f(x), x \in [a, b]$	$L = \int_a^b \sqrt{1 + (y')^2}\, dx$	$y' = \frac{dy}{dx}$
$x = g(y), y \in [c, d]$	$L = \int_c^d \sqrt{1 + (x')^2}\, dy$	$x' = \frac{dx}{dy}$
Parametric: $x(t), y(t)$	$L = \int_\alpha^\beta \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$	$\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$
Polar: $r = f(\theta)$	$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (r')^2}\, d\theta$	$r' = \frac{dr}{d\theta}$

Summary of Topics 7–13

Topic	Core Concept	Key Formula
7. Limit of a Sum	Integration defined as limit	$\int_a^b f\, dx = \lim_{n\to\infty} \sum f(x_i)\Delta x$
8. FTIC	Differentiation ↔ Integration	$\frac{d}{dx}\int_a^x f\, dt = f(x)$
9. MVT for Integrals	Average value of a function	$f(c) = \frac{1}{b-a}\int_a^b f\, dx$
10. Reduction Formulae	Reduce I_n to I_{n-2}	$I_n = \frac{n-1}{n} I_{n-2}$ (for $\sin^n$, $\cos^n$)

Topic	Core Concept	Key Formula
11. Improper Integrals	Limits for infinite/singular integrands	$\int_a^\infty f\,dx = \lim_{t\to\infty}\int_a^t f\,dx$
12. Tests of Convergence	Determine if integral is finite	p-Test, Comparison, Limit Comparison
13. Rectification	Arc length of a curve	$L = \int_a^b \sqrt{1+(y')^2}\,dx$

These seven topics together form the complete toolkit for integration, moving from foundational definitions through advanced techniques to geometric applications.