Chapter 2: Integral Calculus

Tutorial

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Unit 2: Tutorial

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Course: Calculus and Ordinary Differential Equations
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Problem Sets

1. Integration as the Limit of a Sum

Problem 1.1: Using the definition of integration as the limit of a sum, evaluate $\int_0^1 x^2\,dx$ by dividing $[0,1]$ into n equal subintervals and taking the limit as $n \to \infty$.

Problem 1.2: Express $\int_1^4 (2x+3)\,dx$ as the limit of a Riemann sum using right endpoints and n equal subintervals. Evaluate the limit.

Problem 1.3: Use the limit of a sum to find $\int_0^2 e^x\,dx$ by partitioning $[0,2]$ into n equal parts.

Problem 1.4: Evaluate $\int_0^\pi \sin x\,dx$ by expressing it as $\lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x$ where $\Delta x = \frac{\pi}{n}$.

Problem 1.5: A rocket's velocity increases according to $v(t) = 50t$ m/s during the first 10 seconds of launch. Using the limit of a sum with n equal time intervals, derive the formula for total distance traveled (hint: distance = $\int_0^{10} v(t)\,dt$). Verify your result represents the area under the velocity-time curve, which is fundamental in flight trajectory calculations.

2. Fundamental Theorem of Integral Calculus

Problem 2.1: If $F(x) = \int_0^x t^3\,dt$, find $F'(x)$ using the Fundamental Theorem of Calculus.

Problem 2.2: Evaluate $\int_1^3 (x^2 - 4x + 3)\,dx$ using the Fundamental Theorem.

Problem 2.3: Find $\frac{d}{dx}\int_0^{x^2} \cos(t^2)\,dt$ using the chain rule and FTIC.

Problem 2.4: Given $G(x) = \int_{\sin x}^{x^2} e^{t^2}\,dt$, find $G'(x)$.

Problem 2.5: In electrical engineering, the charge $Q(t)$ accumulated on a capacitor is related to current $I(t)$ by $Q(t) = \int_0^t I(\tau)\,d\tau$. If the current is $I(t) = 5\sin(100\pi t)$ amperes, find the total charge accumulated from $t = 0$ to $t = 0.01$ seconds. Use FTIC to also determine the instantaneous rate of charge accumulation at any time t.

3. Mean Value Theorems for Integrals

Problem 3.1: Verify the First Mean Value Theorem for $f(x) = x^2$ on $[1,3]$ by finding the value c such that $\int_1^3 x^2\,dx = f(c)\cdot(3-1)$.

Problem 3.2: If $f(x) = \sin x$ on $[0,\pi]$, find the value guaranteed by the Mean Value Theorem for Integrals such that the area under the curve equals $f(c)\cdot\pi$.

Problem 3.3: Use the Second Mean Value Theorem for Integrals to show that there exists $c \in [0, 2]$ such that $\int_0^2 xe^x \, dx = e^c \int_0^2 x \, dx$.

Problem 3.4: For $f(x) = \frac{1}{x}$ on $[1, e]$, find the average value of the function and the point where this average is attained.

Problem 3.5: A solar panel's power output varies throughout the day according to $P(t) = 300 \sin\left(\frac{\pi t}{12}\right)$ watts for $0 \leq t \leq 12$ hours. The total energy produced is $E = \int_0^{12} P(t) \, dt$. Use the Mean Value Theorem for Integrals to find the time c when the instantaneous power output equals the average power output over the 12-hour period. This helps engineers determine optimal battery charging schedules.

4. Reduction Formulae

Problem 4.1: Derive the reduction formula for $I_n = \int \sin^n x \, dx$ and use it to evaluate $\int \sin^4 x \, dx$.

Problem 4.2: Using the reduction formula for $I_n = \int x^n e^x \, dx$, evaluate $\int x^3 e^x \, dx$.

Problem 4.3: Establish the reduction formula for $I_n = \int \sec^n x \, dx$ and use it to find $\int \sec^4 x \, dx$.

Problem 4.4: For $I_{m,n} = \int \sin^m x \cos^n x \, dx$, derive a reduction formula and apply it to evaluate $\int \sin^3 x \cos^2 x \, dx$.

Problem 4.5: In quantum mechanics, the normalization of wave functions often requires integrals of the form $I_n = \int_0^\infty x^n e^{-ax^2} \, dx$ where $a > 0$. The reduction formula is $I_n = \frac{n-1}{2a} I_{n-2}$ with $I_0 = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ and $I_1 = \frac{1}{2a}$. Use this reduction formula to evaluate I_4 for $a = 1$, which appears in calculating expectation values for the harmonic oscillator ground state.

5. Improper Integrals

Problem 5.1: Evaluate the improper integral $\int_1^\infty \frac{1}{x^2} \, dx$.

Problem 5.2: Determine whether $\int_0^\infty e^{-3x} \, dx$ converges or diverges. If it converges, find its value.

Problem 5.3: Evaluate $\int_0^1 \frac{1}{\sqrt{x}} \, dx$ (improper due to unbounded integrand at $x = 0$).

Problem 5.4: Determine the convergence of $\int_2^\infty \frac{1}{x \ln x} \, dx$.

Problem 5.5: The Laplace transform of a function $f(t)$ is defined as $F(s) = \int_0^\infty e^{-st} f(t) \, dt$, which is an improper integral. Find the Laplace transform of $f(t) = e^{2t}$ by evaluating $\int_0^\infty e^{-st} e^{2t} \, dt$ for $s > 2$. Laplace transforms are essential in control systems engineering for analyzing circuit responses and mechanical vibrations.

6. Tests of Convergence

Problem 6.1: Use the Comparison Test to determine whether $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ converges (compare with $\int_1^{\infty} \frac{1}{x^2}\,dx$).

Problem 6.2: Apply the Limit Comparison Test to determine convergence of $\int_1^{\infty} \frac{x}{x^3+5x+1}\,dx$.

Problem 6.3: Use the p-Test to determine for which values of p the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges.

Problem 6.4: Determine the convergence of $\int_0^{1} \frac{1}{x^p}\,dx$ for different values of p using the p-Test for improper integrals at finite endpoints.

Problem 6.5: In probability theory, the Pareto distribution has probability density function $f(x) = \frac{\alpha x_m^{\alpha}}{x^{\alpha+1}}$ for $x \geq x_m > 0$. For the distribution to be valid, we need $\int_{x_m}^{\infty} f(x)\,dx = 1$. Use the p-Test to determine for which values of α this integral converges, then find the specific value that makes it equal to 1. This distribution models wealth distribution and is used in risk analysis for tech startups.

7. Rectification (Arc Length)

Problem 7.1: Find the arc length of the curve $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 3$.

Problem 7.2: Calculate the length of the arc of the parabola $y^2 = 4ax$ from $x = 0$ to $x = a$.

Problem 7.3: Find the arc length of the curve $y = \ln(\sec x)$ from $x = 0$ to $x = \frac{\pi}{4}$.

Problem 7.4: Determine the length of one arch of the cycloid given parametrically by $x = a(t - \sin t)$, $y = a(1 - \cos t)$ for $0 \leq t \leq 2\pi$.

Problem 7.5: A fiber optic cable is laid along a path modeled by the catenary curve $y = 10\cosh\left(\frac{x}{10}\right)$ meters, where $-20 \leq x \leq 20$ meters. Calculate the total length of cable needed using the arc length formula $L = \int_{-20}^{20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. (Note: $\frac{d}{dx}\cosh(x) = \sinh(x)$ and $\cosh^2(x) - \sinh^2(x) = 1$). Accurate cable length calculation is critical for cost estimation and signal loss prediction in telecommunications infrastructure.
