Chapter 3: Multiple Integrals

Lecture Notes

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Unit 3: Lecture Notes

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Contents

Detailed Notes: Multiple Integrals

Topic 14: Double Integrals

14.1 What is a Double Integral?

A **single integral** $\int_a^b f(x)\,dx$ adds up infinitely many thin strips of area under a curve in 2D.

A **double integral** $\iint_R f(x,y)\,dA$ extends this idea to **3D** — it adds up infinitely many thin columns of volume above a region R in the xy-plane.

Physical meaning: - If $f(x,y) = 1$: the double integral gives the **area** of region R - If $f(x,y) \geq 0$: the double integral gives the **volume** under the surface $z = f(x,y)$ above region R - If $f(x,y) = \rho(x,y)$ (density): the double integral gives the **total mass** of the region

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14.2 Iterated Integrals — The Key to Computation

We never actually "add up tiny columns" directly. Instead, we convert to an **iterated integral** — integrating one variable at a time.

For a rectangular region $R = [a,b] \times [c,d]$:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini's Theorem: For continuous f on a rectangle, both orders of integration give the same result.

14.3 Double Integrals over Non-Rectangular Regions

Type I Region (vertical slices): $R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II Region (horizontal slices): $R = \{(x,y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

14.4 How to Determine Limits of Integration

Step 1: Sketch the region R.

Step 2: For Type I (integrate y first): - x ranges from left boundary to right boundary (constants a to b) - For fixed x, y ranges from the bottom curve $g_1(x)$ to top curve $g_2(x)$

Step 3: For Type II (integrate x first): - y ranges from bottom to top (constants c to d) - For fixed y, x ranges from left curve $h_1(y)$ to right curve $h_2(y)$

Example 14.1 — Rectangle, Both Orders

Problem: Evaluate $\iint_R (3x^2 + 2y)\, dA$ over the rectangle $R = [0, 2] \times [1, 3]$.

Solution:

Integrating y first, then x:

$$I = \int_0^2 \int_1^3 (3x^2 + 2y)\, dy\, dx$$

Inner integral (treat x as constant):

$$\int_1^3 (3x^2 + 2y)\, dy = \Big[3x^2 y + y^2\Big]_1^3 = (9x^2 + 9) - (3x^2 + 1) = 6x^2 + 8$$

Outer integral:

$$I = \int_0^2 (6x^2 + 8)\, dx = \Big[2x^3 + 8x\Big]_0^2 = (16 + 16) - 0 = \mathbf{32}$$

Verification — integrating x first, then y:

$$I = \int_1^3 \int_0^2 (3x^2 + 2y)\, dx\, dy$$

Inner integral:

$$\int_0^2 (3x^2 + 2y)\, dx = \Big[x^3 + 2xy\Big]_0^2 = 8 + 4y$$

Outer integral:

$$I = \int_1^3 (8 + 4y)\, dy = \Big[8y + 2y^2\Big]_1^3 = (24 + 18) - (8 + 2) = 42 - 10 = \mathbf{32} \checkmark$$

Example 14.2 — Triangular Region (Type I)

Problem: Evaluate $\iint_R xy\, dA$ where R is the triangular region bounded by $x = 0$, $y = 0$, and $x + y = 1$.

Solution:

Step 1: Sketch the region

The triangle has vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. The line $x + y = 1$ gives $y = 1 - x$.

Step 2: Set up limits (Type I)

- x: from 0 to 1
- For fixed x, y: from 0 to $1 - x$

$$I = \int_0^1 \int_0^{1-x} xy\, dy\, dx$$

Step 3: Inner integral

$$\int_0^{1-x} xy\, dy = x \left[\frac{y^2}{2}\right]_0^{1-x} = x \cdot \frac{(1-x)^2}{2} = \frac{x(1-x)^2}{2}$$

Step 4: Outer integral

$$I = \int_0^1 \frac{x(1-x)^2}{2}\, dx = \frac{1}{2} \int_0^1 x(1 - 2x + x^2)\, dx$$

$$= \frac{1}{2} \int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{6-8+3}{12}\right) = \frac{1}{2}\cdot\frac{1}{12} = \boxed{\frac{1}{24}}$$

Example 14.3 — Parabolic Region (Type I)

Problem: Evaluate $\iint_R (x+y)\,dA$ where R is bounded by $y = x^2$ and $y = 2x$.

Solution:

Step 1: Find intersection points

$x^2 = 2x \implies x^2 - 2x = 0 \implies x(x-2) = 0 \implies x = 0$ or $x = 2$

Intersection points: $(0,0)$ and $(2,4)$.

Step 2: Identify upper and lower curves

For $0 \le x \le 2$: line $y = 2x$ is above parabola $y = x^2$.

Step 3: Set up integral (Type I)

$$I = \int_0^2 \int_{x^2}^{2x} (x+y)\,dy\,dx$$

Step 4: Inner integral

$$\int_{x^2}^{2x} (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_{x^2}^{2x}$$

$$= (2x^2 + 2x^2) - \left(x^3 + \frac{x^4}{2}\right) = 4x^2 - x^3 - \frac{x^4}{2}$$

Step 5: Outer integral

$$I = \int_0^2 \left(4x^2 - x^3 - \frac{x^4}{2}\right) dx = \left[\frac{4x^3}{3} - \frac{x^4}{4} - \frac{x^5}{10}\right]_0^2$$

$$= \frac{32}{3} - 4 - \frac{32}{10} = \frac{32}{3} - 4 - \frac{16}{5}$$

$$= \frac{160 - 60 - 48}{15} = \frac{52}{15}$$

$$\boxed{I = \frac{52}{15}}$$

Example 14.4 — Type II Region

Problem: Evaluate $\iint_R y^2 \, dA$ where R is bounded by $x = y^2$ and $x = y + 2$.

Solution:

Step 1: Find intersections

$y^2 = y + 2 \implies y^2 - y - 2 = 0 \implies (y-2)(y+1) = 0$

$y = -1$ or $y = 2$. Points: $(1, -1)$ and $(4, 2)$.

Step 2: Set up as Type II (integrate x first — natural since boundaries are x as functions of y)

For $-1 \le y \le 2$: x goes from y^2 (left) to $y + 2$ (right).

$$I = \int_{-1}^{2} \int_{y^2}^{y+2} y^2 \, dx \, dy$$

Step 3: Inner integral

$$\int_{y^2}^{y+2} y^2 \, dx = y^2 \Big[x \Big]_{y^2}^{y+2} = y^2(y + 2 - y^2)$$

Step 4: Outer integral

$$I = \int_{-1}^{2} y^2(y + 2 - y^2) \, dy = \int_{-1}^{2} (y^3 + 2y^2 - y^4) \, dy$$

$$= \left[\frac{y^4}{4} + \frac{2y^3}{3} - \frac{y^5}{5} \right]_{-1}^{2}$$

At $y = 2$: $4 + \frac{16}{3} - \frac{32}{5} = \frac{60+80-96}{15} = \frac{44}{15}$

At $y = -1$: $\frac{1}{4} - \frac{2}{3} + \frac{1}{5} = \frac{15-40+12}{60} = \frac{-13}{60}$

$$I = \frac{44}{15} + \frac{13}{60} = \frac{176+13}{60} = \boxed{\frac{189}{60} = \frac{63}{20}}$$

Example 14.5 — e^{x^2} Type (Requires Switching Order)

Problem: Evaluate $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$.

Note: $\int e^{y^2}\,dy$ has no closed form when integrated with respect to y as the outer integral — we must switch order.

Step 1: Identify the region

The region is: $0 \le x \le 1$, $x \le y \le 1$.

This means: $0 \le x \le y$ for fixed y, and $0 \le y \le 1$.

Step 2: Switch to Type II

$$I = \int_0^1 \int_0^y e^{y^2}\,dx\,dy$$

Step 3: Inner integral

$$\int_0^y e^{y^2}\,dx = e^{y^2}\Big[x\Big]_0^y = ye^{y^2}$$

Step 4: Outer integral

$$I = \int_0^1 ye^{y^2}\,dy = \left[\frac{e^{y^2}}{2}\right]_0^1 = \frac{e-1}{2}$$

$$\boxed{I = \frac{e-1}{2} \approx 0.859}$$

Topic 15: Change of Order of Integration

15.1 What is Change of Order of Integration?

When we write a double integral as $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$, we first integrate with respect to y (inner), then with respect to x (outer).

Change of order means reversing this: we re-write the same integral as $\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, integrating x first, then y.

The value of the integral does not change — only the order of integration changes.

15.2 Why Does This Matter? — The Core Motivation

Changing the order of integration is one of the most powerful techniques in calculus. Here are the key reasons why it matters:

Reason 1: Some integrals are IMPOSSIBLE in the given order

Consider $\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy$. The inner integral $\int \sin(x^2)\,dx$ has no elementary antiderivative. If we switch the order, it becomes $\int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 x\sin(x^2)\,dx$, which is easily solved by substitution!

Reason 2: Sometimes the other order is MUCH simpler

Even when both orders work, one may require much longer computation. Switching can turn a 3-step problem into a 1-step problem.

Reason 3: Used in proving mathematical theorems

Many important results in analysis, probability, and engineering are proved by switching integration order.

Reason 4: Essential in Laplace transforms, Fourier analysis, and probability

Convolution integrals, joint probability distributions, and signal processing all rely on changing the order of integration.

15.3 The KEY Skill: Visualising the Region

The entire process of changing order relies on correctly identifying the region R from the given limits and re-describing it from the opposite perspective.

Golden Rule: > Always sketch the region first. The region R is fixed. Only your description of it changes.

15.4 How to Change the Order — Step-by-Step Procedure

Step 1: Write down the given limits clearly.

Step 2: Identify what region these limits describe. State it in plain language.

Step 3: Sketch the region carefully — draw all boundary curves, shade the region.

Step 4: Re-describe the region using the reversed order of integration: - Find the new constant limits (overall range of the second variable) - Find the new variable limits (range of the first variable for fixed second)

Step 5: Rewrite the integral with the new limits.

Step 6: Evaluate.

Example 15.1 — Classic Switch

Problem: Change the order of integration and evaluate:

$$I = \int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$$

Step 1: Identify the region from given limits

Given: outer $0 \le y \le 1$; inner $y \le x \le 1$.

So the region is: $\{(x, y) : 0 \le y \le 1,\ y \le x \le 1\}$.

Step 2: Re-describe the region

The condition $y \le x$ with $0 \le x \le 1$ means: for a fixed $x \in [0, 1]$, y runs from 0 to x.

So: $\{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}$

Step 3: Rewrite the integral

$$I = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx$$

Step 4: Evaluate

Inner integral (x is constant):

$$\int_0^x \sin(x^2)\,dy = \sin(x^2)\Big[y\Big]_0^x = x\sin(x^2)$$

Outer integral (substitution $t = x^2$, $dt = 2x\,dx$):

$$I = \int_0^1 x\sin(x^2)\,dx = \frac{1}{2}\int_0^1 \sin(t)\,dt = \frac{1}{2}\Big[-\cos t\Big]_0^1 = \frac{1-\cos 1}{2}$$

$$\boxed{I = \frac{1-\cos 1}{2} \approx 0.230}$$

Without changing order: $\int \sin(x^2)\,dx$ has no closed form. The problem would be impossible!

Example 15.2 — Triangular Region

Problem: Change the order and evaluate:

$$I = \int_0^2 \int_0^{2-x} f(x,y)\,dy\,dx \quad \text{(sketch and describe region)}$$

Then evaluate for $f(x,y) = x + y$.

Step 1: Region from given limits

$0 \le x \le 2$; $0 \le y \le 2 - x$.

The boundary $y = 2 - x$ is the line $x + y = 2$.

Region: triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$.

Step 2: Re-describe

For fixed $y \in [0, 2]$: x runs from 0 to $2 - y$.

New integral:

$$I = \int_0^2 \int_0^{2-y} (x + y)\, dx\, dy$$

Step 3: Evaluate

Inner integral:

$$\int_0^{2-y} (x + y)\, dx = \left[\frac{x^2}{2} + xy\right]_0^{2-y} = \frac{(2-y)^2}{2} + y(2-y)$$

$$= \frac{4 - 4y + y^2}{2} + 2y - y^2 = 2 - 2y + \frac{y^2}{2} + 2y - y^2 = 2 - \frac{y^2}{2}$$

Outer integral:

$$I = \int_0^2 \left(2 - \frac{y^2}{2}\right) dy = \left[2y - \frac{y^3}{6}\right]_0^2 = 4 - \frac{8}{6} = 4 - \frac{4}{3} = \boxed{\frac{8}{3}}$$

Example 15.3 — Parabolic Boundary

Problem: Change the order of integration:

$$I = \int_0^4 \int_{\sqrt{x}}^2 f(x, y)\, dy\, dx$$

Step 1: Identify region

$0 \le x \le 4$; $\sqrt{x} \le y \le 2$.

Boundaries: $y = \sqrt{x}$ (i.e., $x = y^2$) and $y = 2$.

Region: area between $y = \sqrt{x}$ and $y = 2$ for $x \in [0, 4]$.

Step 2: Re-describe with y as outer variable

y ranges from 0 to 2 (since $\sqrt{x}$ ranges over $[0, 2]$ for $x \in [0, 4]$).

For fixed $y \in [0, 2]$: $y = \sqrt{x}$ gives $x = y^2$. So x runs from 0 to y^2.

$$\boxed{I = \int_0^2 \int_0^{y^2} f(x,y)\,dx\,dy}$$

Example 15.4 — Impossible Inner Integral Resolved

Problem: Evaluate $I = \int_0^2 \int_{x/2}^1 e^{y^2}\,dy\,dx$ by changing the order of integration.

Step 1: Identify region

$0 \leq x \leq 2$; $\frac{x}{2} \leq y \leq 1$.

Boundary: $y = x/2$ means $x = 2y$.

Region: $\{(x,y) : 0 \leq x \leq 2,\ x/2 \leq y \leq 1\}$.

Step 2: Re-describe

y ranges from 0 to 1.

For fixed y: $y = x/2 \Rightarrow x = 2y$. So x goes from 0 to $2y$.

$$I = \int_0^1 \int_0^{2y} e^{y^2}\,dx\,dy$$

Step 3: Evaluate

Inner integral:

$$\int_0^{2y} e^{y^2}\,dx = e^{y^2} \cdot 2y$$

Outer integral:

$$I = \int_0^1 2ye^{y^2}\,dy = \left[e^{y^2}\right]_0^1 = e - 1$$

$$\boxed{I = e - 1 \approx 1.718}$$

Example 15.5 — Three-Region Split

Problem: Change the order of integration for:

$$I = \int_0^4 \int_0^{\sqrt{4x-x^2}} f(x,y)\,dy\,dx$$

Step 1: Identify the boundary

$y = \sqrt{4x - x^2}$ means $y^2 = 4x - x^2 = -(x^2 - 4x) = -(x-2)^2 + 4$

So: $(x-2)^2 + y^2 = 4$ — this is a **circle** of radius 2 centred at $(2, 0)$.

The upper semicircle: $y \geq 0, 0 \leq x \leq 4$.

Step 2: Re-describe

y ranges from 0 to 2.

For fixed y: $(x-2)^2 = 4 - y^2 \Rightarrow x = 2 \pm \sqrt{4 - y^2}$.

So x ranges from $2 - \sqrt{4-y^2}$ to $2 + \sqrt{4-y^2}$.

$$\boxed{I = \int_0^2 \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} f(x,y)\,dx\,dy}$$

Real-Life Application of Change of Order: Signal Processing

Context: In communication engineering, the output of a linear system with input $f(t)$ and impulse response $h(t)$ is the convolution:

$$y(t) = \int_0^t f(\tau)h(t-\tau)\,d\tau$$

To find the total energy of the output signal up to time T:

$$E = \int_0^T y(t)\,dt = \int_0^T \int_0^t f(\tau)h(t-\tau)\,d\tau\,dt$$

This double integral has **variable limits** that make it hard to evaluate directly. By changing the order of integration — noting the region is the triangle $0 \leq \tau \leq t \leq T$ — we rewrite as:

$$E = \int_0^T \int_\tau^T h(t-\tau)\,dt\,f(\tau)\,d\tau$$

This form separates $f(\tau)$ from the inner integral, making evaluation far easier. **This is exactly how engineers compute signal energy and filter responses in DSP systems.**

15.5 Common Mistakes to Avoid

Mistake	Correct Approach
Changing limits without sketching the region	Always draw the region first
Swapping limits without swapping the integration order symbol	Change both the limits AND the $dx\,dy$ order
Not finding the correct new upper/lower bounds	For each fixed value of the new outer variable, trace the region horizontally/vertically
Assuming the answer changes after switching	The integral value is the same — only the order changes

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Topic 16: Triple Integrals

16.1 Introduction

A **triple integral** $\iiint_E f(x,y,z)\,dV$ extends double integration to three dimensions. It "sums" the function f over a three-dimensional solid region E.

Physical interpretations: - If $f = 1$: gives the **volume** of E - If $f = \rho(x,y,z)$ (density): gives the **total mass** - If $f = \rho \cdot z$: contributes to finding the **centre of mass**

16.2 Iterated Triple Integrals

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

There are **six possible orders** of integration: $dz\,dy\,dx$, $dz\,dx\,dy$, $dy\,dz\,dx$, $dy\,dx\,dz$, $dx\,dy\,dz$, $dx\,dz\,dy$.

Setting up limits: 1. Fix x and y, find z-limits (functions of x and y) 2. Fix x, find y-limits (functions of x) 3. Find x-limits (constants)

Example 16.1 — Box Region

Problem: Evaluate $\iiint_E xyz\,dV$ where $E = [0,1] \times [0,2] \times [0,3]$.

Solution:

$$I = \int_0^1 \int_0^2 \int_0^3 xyz\,dz\,dy\,dx$$

Innermost integral (integrate z, treat x, y as constants):

$$\int_0^3 xyz\,dz = xy\left[\frac{z^2}{2}\right]_0^3 = xy \cdot \frac{9}{2} = \frac{9xy}{2}$$

Middle integral (integrate y, treat x as constant):

$$\int_0^2 \frac{9xy}{2}\,dy = \frac{9x}{2}\left[\frac{y^2}{2}\right]_0^2 = \frac{9x}{2} \cdot 2 = 9x$$

Outermost integral:

$$I = \int_0^1 9x\,dx = 9\left[\frac{x^2}{2}\right]_0^1 = \frac{9}{2}$$

$$\boxed{I = \frac{9}{2}}$$

Example 16.2 — Tetrahedron

Problem: Evaluate $\iiint_E z\,dV$ where E is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Solution:

Step 1: Set up limits

The bounding plane is $x + y + z = 1$ (passes through the three axis intercepts).

- x: from 0 to 1
- For fixed x, y: from 0 to $1 - x$
- For fixed x and y, z: from 0 to $1 - x - y$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

Step 2: Inner integral

$$\int_0^{1-x-y} z\,dz = \left[\frac{z^2}{2}\right]_0^{1-x-y} = \frac{(1-x-y)^2}{2}$$

Step 3: Middle integral

$$\int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy$$

Let $u = 1 - x - y$, $du = -dy$. When $y = 0$: $u = 1 - x$; when $y = 1 - x$: $u = 0$.

$$= \frac{1}{2}\int_0^{1-x} u^2\,du = \frac{1}{2}\left[\frac{u^3}{3}\right]_0^{1-x} = \frac{(1-x)^3}{6}$$

Step 4: Outer integral

$$I = \int_0^1 \frac{(1-x)^3}{6}\,dx$$

Let $u = 1 - x$, $du = -dx$:

$$= \frac{1}{6}\int_0^1 u^3\,du = \frac{1}{6}\cdot\frac{1}{4} = \boxed{\frac{1}{24}}$$

Topic 17: Change of Variables — Jacobians of Transformations (Further Reading)

17.1 Why Do We Change Variables?

When a region or integrand is complicated in Cartesian coordinates (x, y) or (x, y, z), a **change of variables** can: 1. Simplify the **region** of integration (e.g., transform an ellipse to a circle) 2. Simplify the **integrand** (e.g., $x^2 + y^2$ becomes r^2) 3. Make an **impossible integral** tractable

Just as substitution $u = g(x)$ in single integrals requires replacing dx with $g'(x)^{-1}du$ (or $dx = \frac{1}{g'(x)}du$), in multiple integrals we need to account for how the transformation **stretches or compresses** areas/volumes.

This scaling factor is the **Jacobian**.

17.2 The Jacobian for 2D Transformations

Given: transformation $x = x(u, v)$, $y = y(u, v)$.

The **Jacobian of the transformation** is the determinant:

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Change of variables formula:

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v),\, y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

where S is the transformed region in the uv-plane.

17.3 The Jacobian for 3D Transformations

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\iiint_E f(x,y,z)\,dV = \iiint_S f \cdot |J|\,du\,dv\,dw$$

Real-Life Application of Jacobians: GPS and Coordinate Transformations

Context: A GPS receiver receives signals and computes your position in **spherical coordinates** (latitude ϕ, longitude λ, altitude ρ) from satellites. To compute distances, areas, or to integrate quantities over geographic regions (like total rainfall over a watershed), engineers must convert to Cartesian coordinates.

The Jacobian of the spherical-to-Cartesian transformation is $\rho^2 \sin\phi$. When computing the **total amount of a resource** (e.g., CO□ absorbed) over a spherical cap region of Earth's atmosphere:

$$\text{Total CO}_2 = \iiint_{\text{region}} C(\rho,\phi,\lambda) \cdot \rho^2 \sin\phi\,d\rho\,d\phi\,d\lambda$$

Without the Jacobian $\rho^2 \sin\phi$, the computed quantities would be completely wrong — areas near the poles are physically smaller but their angular extent is the same. The Jacobian accounts for this **geometric distortion**.

This is used in: - Climate modelling (integrating temperature/humidity over atmosphere) - Satellite image analysis - Navigation systems - Astronomy (computing flux from extended sources)

Topic 18: Applications — Computations of Areas and Volumes

18.1 Area Using Double Integrals

Concept: The area of a region R in the xy-plane is:

$$A = \iint_R dA = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

Why use double integrals for area? - It is consistent with the general multiple integration framework - It extends naturally to polar, parametric, and curvilinear coordinates - When $f(x, y) = 1$, the double integral literally counts "how many unit squares fit in R"

18.2 Volume Using Double and Triple Integrals

Volume between two surfaces $z = f_1(x, y)$ (bottom) and $z = f_2(x, y)$ (top):

$$V = \iint_R [f_2(x, y) - f_1(x, y)] \, dA$$

Volume of a solid E directly:

$$V = \iiint_E dV$$

Example 18.1 — Area Enclosed by Curves

Problem: Find the area of the region bounded by the parabola $y = x^2$ and the line $y = x + 2$.

Concept in real life: An engineer designing a highway overpass needs to compute the cross-sectional area between a parabolic arch and a flat roadbed to estimate material volumes for formwork.

Solution:

Step 1: Find intersection points

$x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0$

$x = -1$ or $x = 2$. Points: $(-1, 1)$ and $(2, 4)$.

Step 2: Identify top and bottom curves

For $-1 \leq x \leq 2$: line $y = x + 2$ is above parabola $y = x^2$.

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy \, dx$$

Step 3: Inner integral

$$\int_{x^2}^{x+2} dy = (x+2) - x^2$$

Step 4: Outer integral

$$A = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

At $x = 2$: $2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{10}{3}$

At $x = -1$: $\frac{1}{2} - 2 + \frac{1}{3} = \frac{3-12+2}{6} = \frac{-7}{6}$

$$A = \frac{10}{3} + \frac{7}{6} = \frac{20}{6} + \frac{7}{6} = \frac{27}{6} = \boxed{\frac{9}{2}}$$

Answer: The enclosed area is $\frac{9}{2}$ square units.

Example 18.2 — Volume of a Solid

Problem: Find the volume of the solid bounded above by the paraboloid $z = 4 - x^2 - y^2$ and below by the xy-plane ($z = 0$).

Concept in real life: A structural dome or satellite dish is often parabolic in cross-section. Finding the volume enclosed helps architects calculate the amount of concrete or structural material needed.

Solution:

$$\boxed{V = 8\pi \approx 25.13 \text{ cubic units}}$$

Engineering Interpretation: A concrete dome of this shape with z in meters would require approximately 25.13 m^3 of concrete — from which the mass and cost can be directly computed.

Topic 19: Applications — Centre of Mass and Centroid (Further Reading)

19.1 Concept and Motivation

The centre of mass (or **centroid** for uniform density) is the single point at which the entire mass of an object can be considered to act — like the balance point.

Why it matters: - Structural engineers compute it to ensure buildings and bridges don't topple - Aerospace: the centre of mass of a rocket or aircraft must align with the thrust vector - Manufacturing: finding where to drill a hole for a pivot point

Key Formulas for a 2D Lamina (thin plate)

For a lamina occupying region R with density $\rho(x, y)$:

$$\text{Total mass: } M = \iint_R \rho(x, y)\, dA$$

$$\text{Moment about } y\text{-axis: } M_y = \iint_R x\, \rho(x, y)\, dA$$

$$\text{Moment about } x\text{-axis: } M_x = \iint_R y\, \rho(x, y)\, dA$$

$$\text{Centre of mass: } \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

For uniform density $□= $ constant (centroid):

$$\bar{x} = \frac{1}{A} \iint_R x\, dA, \quad \bar{y} = \frac{1}{A} \iint_R y\, dA$$

Topic 20: Applications — Moment of Inertia (Further Reading)

20.1 Concept and Physical Significance

The **moment of inertia** (also called the **second moment of mass**) measures an object's resistance to **rotational acceleration** — the rotational analogue of mass.

$$\text{Newton's 2nd Law (linear): } F = ma$$

$$\text{Newton's 2nd Law (rotational): } \tau = I\alpha$$

where τ is torque, I is moment of inertia, and α is angular acceleration.

The larger I is, the harder it is to spin the object (or change its spin).

Why does shape matter? Mass far from the rotation axis contributes **much more** to I (since $I \propto r^2$). This is why: - A figure skater pulls arms in to spin faster (reducing I) - Flywheels are made with mass at the rim (maximising I for energy storage) - I-beams in construction have flanges far from the neutral axis (large I for bending resistance)

20.2 Formulas for Moment of Inertia

For a 2D lamina with density $\rho(x, y)$:

$$I_x = \iint_R y^2\, \rho(x, y)\, dA \quad \text{(about } x\text{-axis)}$$

$$I_y = \iint_R x^2\, \rho(x, y)\, dA \quad \text{(about } y\text{-axis)}$$

$$I_0 = \iint_R (x^2 + y^2)\, \rho(x, y)\, dA \quad \text{(polar moment, about origin)}$$

Note: $I_0 = I_x + I_y$

For a 3D solid with density $\rho(x, y, z)$:

$$I_z = \iiint_E (x^2 + y^2)\, \rho\, dV \quad \text{(about } z\text{-axis)}$$

20.3 Summary: Applications at a Glance

Quantity	Formula	Physical Meaning
Area	$A = \iint_R dA$	Space occupied by the region
Volume	$V = \iint_R [f_2 - f_1]\, dA$ or $\iiint_E dV$	Space enclosed by 3D solid

Quantity	Formula	Physical Meaning
Mass	$M = \iint_R \rho \, dA$	Total material in lamina
Centroid $\bar{x}$	$\frac{1}{A} \iint_R x \, dA$	Balance point (uniform density)