Module 2

UNIT-I

MULTIPLE INTEGRALS

INTRODUTION

- When a function $f(x)$ is integrated with respect to *x* between the limits *a* and *b*, we get the double integral $\int_a^b f(x)dx$.
- If the integrand is a function $f(x, y)$and if it is integrated with respect to *x* and *y* repeatedly between the limits x_0 and x_1 (for *x*) and between the limits y_0 and y_1 (for *y*) we get a **double integral** that is denoted by the symbol $\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y)dxdy$.
- Extending the concept of double integral one step further, we get the **triple integral,** denoted by $\int_{z_0}^{z_1} \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z)dxdydz$.

Evaluation of Double and Triple Integrals

- To evaluate $\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy$ first integrate $f(x,y)$ with respect to x partially, treating y as constant temporarily, between the limits x_0 and x_1.
- Then integrate the resulting function of y with respect to y between the limits y_0 and y_1 as usual.
- In notation $\int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} f(x,y)dx\right]dy$ (for double integral)

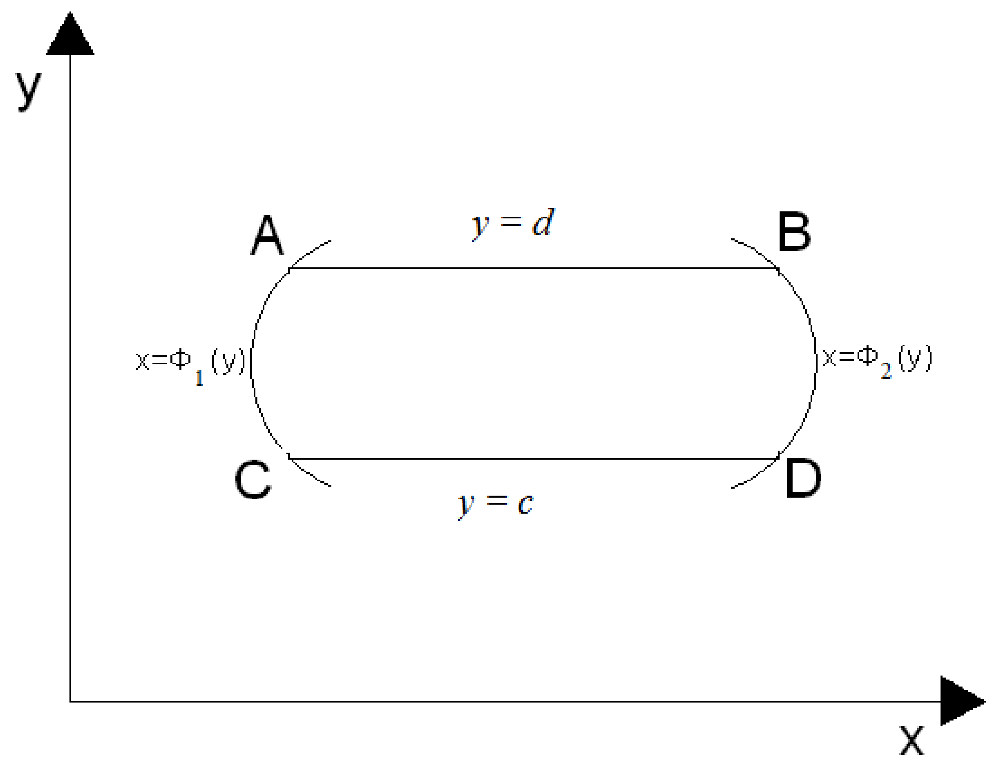
  $\int_{z_0}^{z_1}\left\{\int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} f(x,y,z)dx\right]dy\right\}dz$ (for triple integral).

Note:

- Integral with variable limits should be the innermost integral and it should be integrated first and then the constant limits.

REGION OF INTEGRATION

Consider the double integral $\int_c^d \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dxdy$, x varies from $\varphi_1(y)\ to\ \varphi_2(y)$ and y varies from $c\ to\ d$. (i.e) $\varphi_1(y) \le x \le \varphi_2(y)$and $c \le y \le d$. These inequalities determine a region in the $xy-plane$, which is shown in the following figure.This region ABCD is known as the region of integration



EXAMPLE :1

Evaluate $\int_0^1 \int_0^2 y^2 x \, dy \, dx$

Solution:

$$\int_0^1 \int_0^2 y^2 x \, dy \, dx = \int_0^1 x[y/3]_0^2 \, dx$$

$$= \frac{8}{3} \int_0^1 x \, dx$$

$$= \frac{8}{3} \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{4}{3}$$

EXAMPLE :2

Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\, dy\, dx$

Solution:

$$\int_2^3 \int_1^2 \frac{1}{xy}\, dy\, dx = \int_2^3 [\log x]_1^2 \frac{1}{y}\, dy$$

$$=(log2 - log1) \int_2^3 \frac{1}{y}\, dy$$

$$=log2\ [logy]_2^3$$

$$=log2(log3 - log2)$$

$$=log2.\log(3/2)$$

EXAMPLE :3

Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2 z dz dy dx$

Solution:

$$\int_0^2 \int_1^3 \int_1^2 xy^2 z dz dy dx = \int_0^2 \int_1^3 \left[\frac{z^2}{2}\right]_1^2 xy^2 dy dx$$

$$= \int_0^2 \int_1^3 \frac{3}{2} xy^2 dy dx$$

$$= \frac{3}{2} \int_0^2 \left[\frac{y^3}{3}\right]_1^3 x dx$$

$$= \frac{26}{2} \left[\frac{x^2}{2}\right]_0^2 = 26$$

EXAMPLE :4

Evaluate $\int_0^1 dx \int_0^2 dy \int_1^2 yx^2 z dz$

Solution:

$$\int_0^1 dx \int_0^2 dy \int_1^2 yx^2 z dz = \int_0^1 dx \int_0^2 dy \left[\frac{z^2}{2}\right]_1^2 \ yx^2$$

$$= \frac{3}{2}\int_0^1 \left[\frac{y^2}{2}\right]_0^2 x^2 dx$$

$$= \frac{3}{2}\int_0^1 2\, x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 \ = 1$$

EXAMPLE :5

Evaluate $\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} r^2 \sin\theta \, dr d\theta d\emptyset$

Solution:

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{1} r^2 \sin\theta \, dr d\theta d\emptyset = \int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{r^3}{3}\right]_0^1 d\theta d\emptyset$$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin\theta d\theta d\emptyset$$

$$= \frac{1}{3}\int_0^{\pi}[-cos\theta]_0^{\frac{\pi}{2}} d\emptyset$$

$$= \frac{1}{3}\int_0^{\pi} d\emptyset$$

$$= \frac{\pi}{3}$$

EXAMPLE :6

Evaluate $\int_0^1 \int_0^x dx\, dy$

Solution:

$$\int_0^1 \int_0^x dy\, dx = \int_0^1 [y]_0^x\, dx$$

$$= \int_0^1 x\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

EXAMPLE :7

Evaluate $\int_0^a \int_0^x \int_0^y xyzdxdydz$

Solution:

$$I = \int_0^a \int_0^x [\int_0^y zdz]xydydx$$

$$=\int_0^a \int_0^x \left[\frac{z^2}{2}\right]_0^y xydydx$$

$$= \int_0^a \int_0^x \left[\frac{y^2}{2}\right] xydydx$$

$$=\int_0^a \int_0^x \left[\frac{y^3}{2}\right] dy\ xdx \ = \int_0^a \left[\frac{y^4}{8}\right]_0^x xdx$$

$$= \left[\frac{x^6}{48}\right]_0^a = \frac{a^6}{48}$$

EXAMPLE :8

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$

Solution:

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}}\right)\right]_0^{\sqrt{1-x^2-y^2}} dxdy$$

$$=\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dxdy = \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} dx$$

$$=\frac{\pi}{2} \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x)\right]_0^1$$

$$= \frac{\pi^2}{8}$$

EXAMPLE :9

Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Solution:

$$I = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} a^2 sin^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left[\frac{1-cos\,2\theta}{2}\right] d\theta$$

$$= \frac{a^2}{2} X \frac{1}{2}\left[\theta - \frac{sin\,2\theta}{2}\right]_0^{\pi} = \frac{\pi a^2}{4}$$

PROBLEMS FOR PRACTICE

Evaluate the following

1. $\int_0^2 \int_0^1 4xy \, dxdy$ — Ans: 4

2. $\int_1^b \int_1^a \frac{1}{xy} \, dxdy$ — Ans: loga.logb

3. $\int_0^1 \int_0^x \, dxdy$ — Ans: 1/2

4. $\int_0^\pi \int_0^{\sin\theta} r \, drd\theta$ — Ans: π/4

5. $\int_0^1 \int_0^2 \int_0^3 xyzdxdydz$ — Ans: 9/2

6. $\int_0^1 \int_0^z \int_0^{y+z} dzdydx$ — Ans: ½

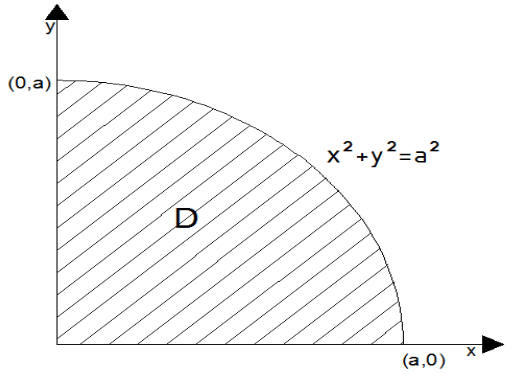
EXAMPLE :10

Sketch the region of integration for $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\, dydx$.

Solution:

Given $x = 0 \; and \; x = a$; $y = 0 \; and \; y^2 = a^2 - x^2$

$$y = 0 \text{ and } x^2 + y^2 = a^2$$

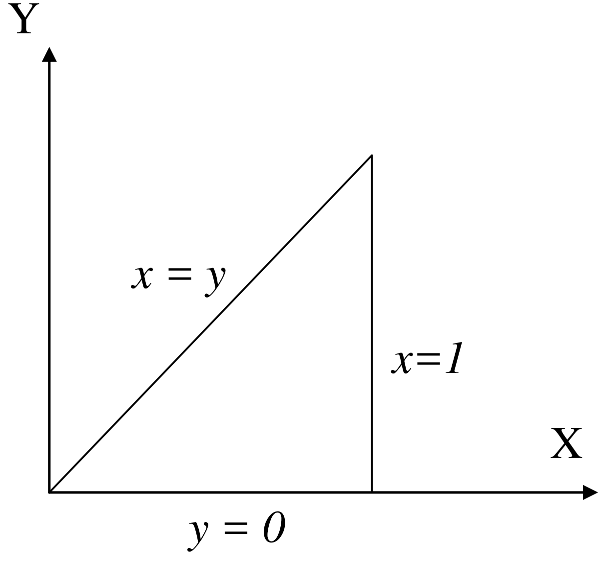


EXAMPLE :11

Sketch the region of integration for $\int_0^1 \int_0^x f(x,y)\, dydx$.

Solution:

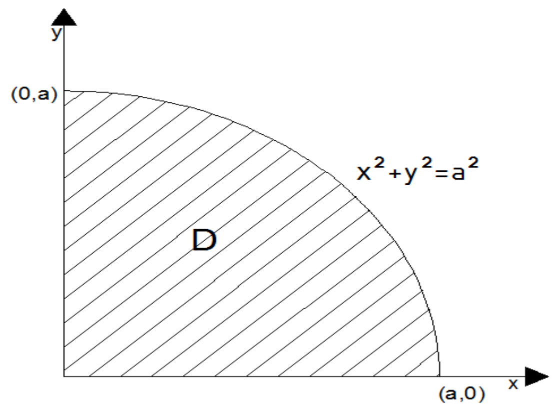
Given $x = 0$; $x = 1$ and $y = 0$; $y = x$.



EXAMPLE :12

Evaluate $\iiint_D \; xyz\,dxdydz$ where D is the region bounded by the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$

Solution:



$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\;dzdydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} \;dydx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,(a^2 - x^2 - y^2)dydx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} x\,(a^2y - yx^2 - y^3)dydx$$

$$= \frac{1}{2}\int_0^a \left[a^2\frac{y^2}{2} - x^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} xdx$$

$$= \frac{1}{8}\int_0^a x\,(a^2 - x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^a (a^4x - 2a^2x^3 + x^5)\;dx$$

$$= \frac{1}{8}\left[a^4\frac{x^2}{2} - 2a^2\frac{x^4}{4} - \frac{x^6}{6}\right]_0^a = \frac{a^6}{48}.$$

PROBLEMS FOR PRACTICE

1.Sketch the region of integration for the following

(i) $\int_0^4 \int_{\frac{y^2}{4}}^{y} \frac{ydxdy}{x^2+y^2}$

(ii) $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y\, dydx$

(iii) $\int_0^1 \int_x^1 \frac{ydxdy}{x^2+y^2}$

2.Evaluate $\iiint_V (xy + yz + zx)dxdydz$, where V is the region of space bounded by *x=0,x=1,y=0,y=2,z=0 and z=3*.

Ans: 33/2

3. Evaluate $\iiint_V \frac{dxdydz}{(1+x+y+z)^3}$, where V is the region of space bounded by *x=0,y=0,z=0 and x+y+z=1*

Ans: $\frac{1}{16}(8log2 - 5)$

4. Evaluate $\iiint_V dxdydz$, where V is the region of space bounded by *x=0,,y=0,,z=0 and 2x+3y+4z=12.*

Ans: 12

CHANGE OF ORDER OF INTEGRATION

- If the limits of integration in a double integral are constants, then the order of integration can be changed, provided the relevant limits are taken for the concerned variables.
- When the limits for inner integration are functions of a variable, the change in the order of integration will result in changes in the limits of integration.

i.e. $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$ will take the form

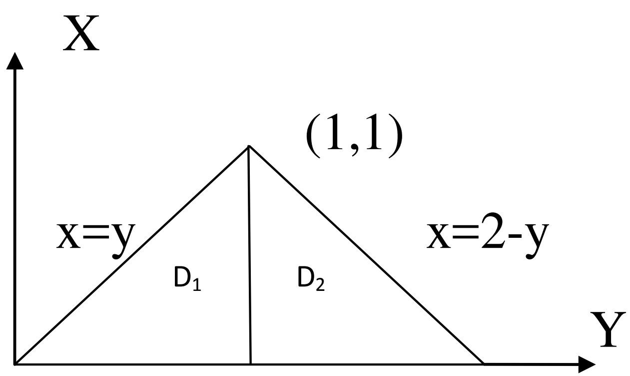
$$\int_a^b \int_{h_1(x)}^{h_2(x)} f(x,y)dydx$$

- This process of converting a given double integral into its equivalent double integral by changing the order of integration is called the **change of order of integration**.

EXAMPLE :13

Evaluate $\int_0^1 \int_y^{2-y} xydx\, dy$ by changing the order of integration.

Solution:



Given y : 0 to 1 and x : y to 2-y

By changing the order of integration,

In Region D_1 x : 0 to 1 and y : 0 to x.

In Region D_2 x : 1 to 2 and y : 0 to 2-x.

$$\int_0^1 \int_y^{2-y} xydx\,dy = \int_0^1 \int_0^x xydy\,dx + \int_1^2 \int_0^{2-x} xydy\,dx$$

$$= \int_0^1 x\left[\frac{y^2}{2}\right]_0^x dx + \int_1^2 x\left[\frac{y^2}{2}\right]_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^3\,dx + \frac{1}{2}\int_1^2 [4x - 4x^2 + x^3]dx$$

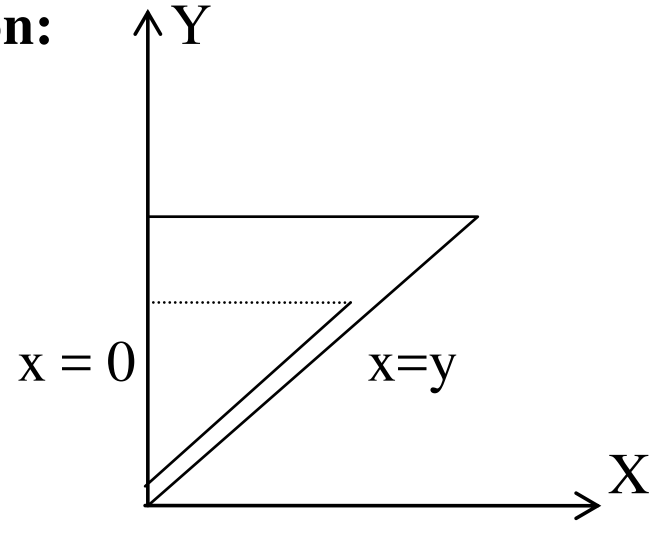
$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[2x^2 - \frac{4x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{8} + \frac{5}{24} = \frac{1}{3}$$

EXAMPLE :14

Evaluate $\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}}dxdy$ by changing the order of integration.

Solution:



Given x=0, x = y, y = 0, y = ∞ .

By changing the order of integration y: x to ∞, x : 0 to ∞

$$\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} dx dy = \int_0^\infty \int_x^\infty y e^{-\frac{y^2}{x}} dy dx$$

$$= \int_0^\infty \int_x^\infty y e^{-\frac{y^2}{x}} d\left(\frac{y^2}{2}\right) dx$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{e^{-\frac{y^2}{x}}}{-1/x} \right]_x^\infty dx \ = \frac{1}{2} \int_0^\infty x e^{-x} dx$$

$Take\ u = x, dv = e^{-x} dx \quad implies\ du = dx, v = -e^{-x},$

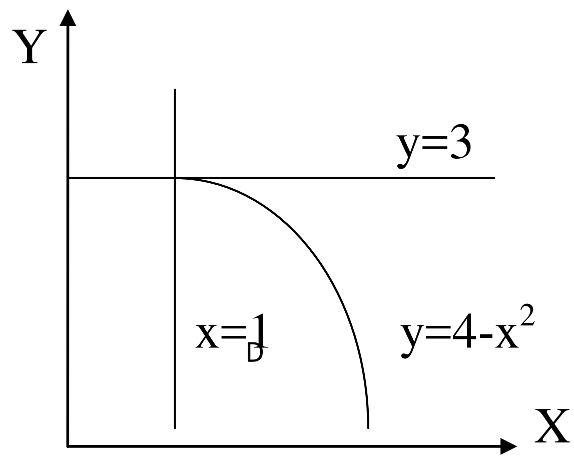
by integration by parts,

$$= \frac{1}{2} \left[x \left(\frac{e^{-x}}{-1} \right) - e^{-x} \right]_0^\infty = \frac{1}{2}$$

EXAMPLE :15

Evaluate $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ by changing the order of integration.

Solution:



Given y=0,y=3 and x=1, x= $\sqrt{4-y}$

By changing the order of integration,

In region D, x : 1 to 2 and y : 0 to $4-x^2$

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy = \int_1^2 \int_0^{4-\mathrm{x}^2} (x+y)\,dy\,dx$$

$$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_0^{4-\mathrm{x}^2} dx$$

$$= \int_1^2 \left[x(4-\mathrm{x}^2) + \frac{(4-\mathrm{x}^2)^2}{2}\right] dx$$

$$= \int_1^2 \left[\frac{x^4}{4} - x^3 - 4x^2 + 4x + 8\right] dx$$

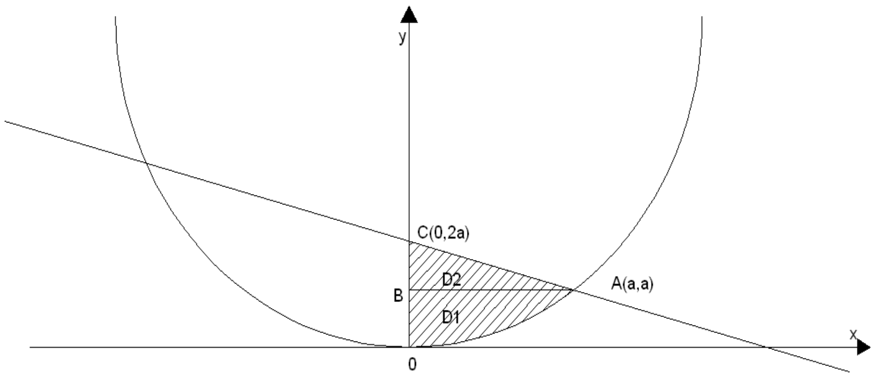
$$= \left[\frac{x^5}{10} - \frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2 + 8x\right]_1^2$$

$$= \frac{241}{8}$$

EXAMPLE :16

Evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx$ by changing the order of integration.

Solution:



Given y : x^2/a to $2a - x$ and x : 0 to a

By changing the order of integration,

In Region D_1 x : 0 to $\sqrt{ay}$ and y : 0 to a.

In Region D_2 x : 0 to $2a - y$ and y : a to 2a.

$$\int_0^a \int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_0^a \int_0^{\sqrt{ay}} xydy\,dx + \int_a^{2a} \int_0^{2a-y} xydy\,dx$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_0^1 y\left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \frac{1}{2}\int_a^{2a}[4a^2y - 4ay^2 + y^3]dy$$

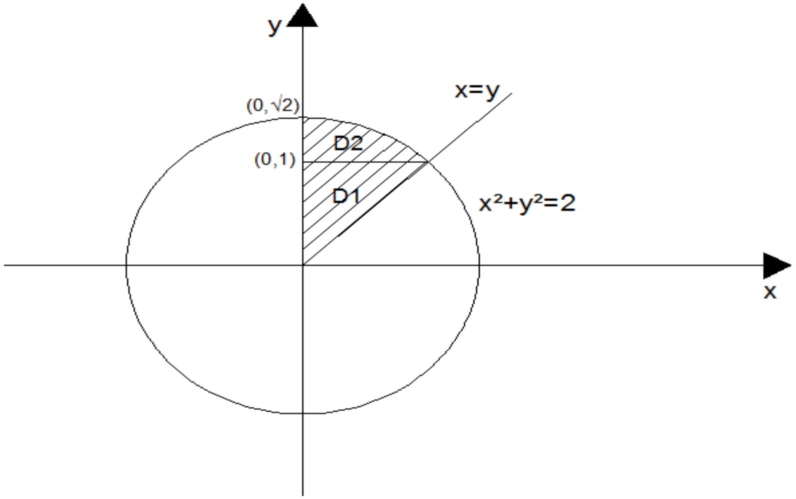
$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{1}{2}\left[2a^2y^2 - \frac{4ay^3}{3} + \frac{y^4}{4}\right]_a^{2a}$$

$$= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8}.$$

EXAMPLE :17

Evaluate $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy\, dx$ by changing the order of integration.

Solution:



Given x = 0, x = 1 and y = x, $y^2 = 2-x^2$

By changing the order of integration

In Region D_1, y : 0 to 1,x : 0 to y

In Region $D_{2,}$ y : 1 to $\sqrt{2}$, x : 0 to $\sqrt{2-y^2}$

$$\mathrm{I} = \int_0^1 \int_0^y \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$= \int_0^1 \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2}} dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left(\sqrt{2}y - y\right)dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y\right)dy$$

$$= \left((\sqrt{2}-1)\frac{y^2}{2}\right)_0^1 + \left(\sqrt{2}y - \frac{y^2}{2}\right)_1^{\sqrt{2}}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

PROBLEMS FOR PRACTICE

Evaluate the following by changing the order of integration

1. $\int_0^a \int_x^a (x^2 + y^2)\, dydx$ Ans: $\frac{a^4}{3}$

2. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dydx$ Ans: $\frac{3a^4}{8}$

3. $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\, dxdy$ Ans: $\frac{a^3}{6}$

4. $\int_0^1 \int_y^{2-y} xy\, dxdy$ Ans: $\frac{1}{3}$

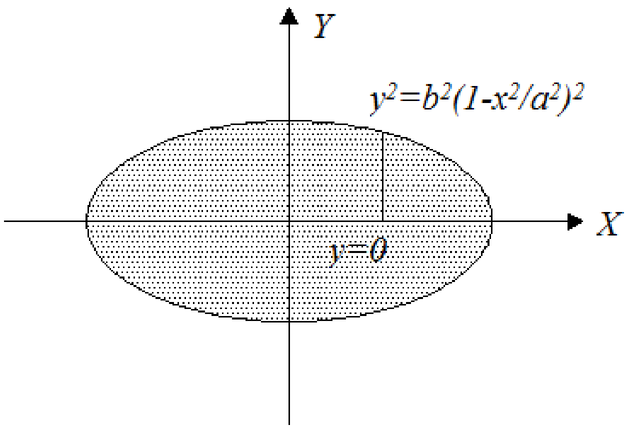
PLANE AREA USING DOUBLE INTEGRAL

CARTESIAN FORM

EXAMPLE :18

Find by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:



$$A = 4\iint dydx = 4\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dydx$$

$$= 4\int_0^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

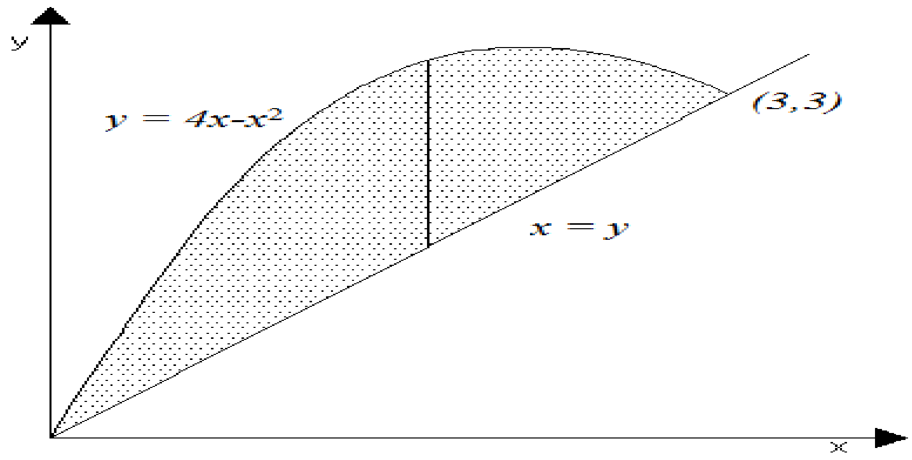
$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a} \text{x} \frac{a^2}{2} \text{x} \frac{\pi}{2} = \pi ab \text{ sq.units.}$$

EXAMPLE :19

Find the area between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution:



Given $y = 4x - x^2$ *and* $y = x$, solving for x,

$$x = 4x - x^2 \;\; => \;\; 0 = 3x - x^2 => \;\; 0 = (3 - x)x => x = 0,3$$

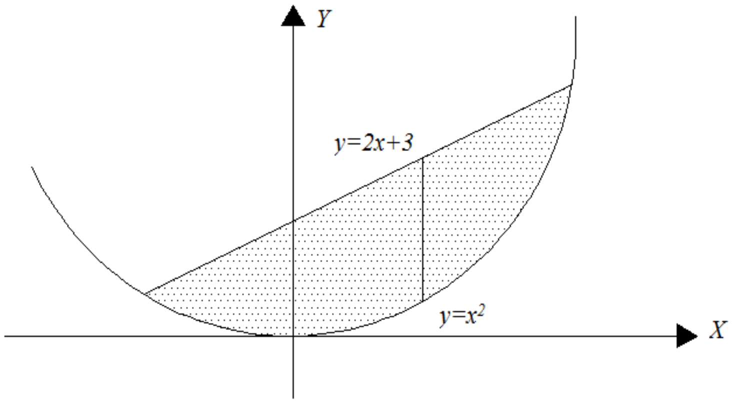
A = $\int_0^3 \int_x^{4x-x^2} dydx = \int_0^3 [y]_x^{4x-x^2} dx$

$= \int_0^3 (3x - x^2) dx$

$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{9}{2}$

EXAMPLE :20

Find the area between the parabola $y = x^2$ and the line $y = 2x + 3$.



Solution:

Given $y = x^2$ and $y = 2x + 3$.

solving for x, $x^2 = 2x + 3 \Rightarrow x = -1,3$

$$A = \int_{-1}^{3}\int_{x^2}^{2x+3} dydx = \int_{-1}^{3}[y]_{x^2}^{2x+3}dx$$

$$= \int_{-1}^{3}(2x + 3 - x^2)dx$$

$$= \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3} = \frac{32}{3}$$

PLANE AREA USING DOUBLE INTEGRAL

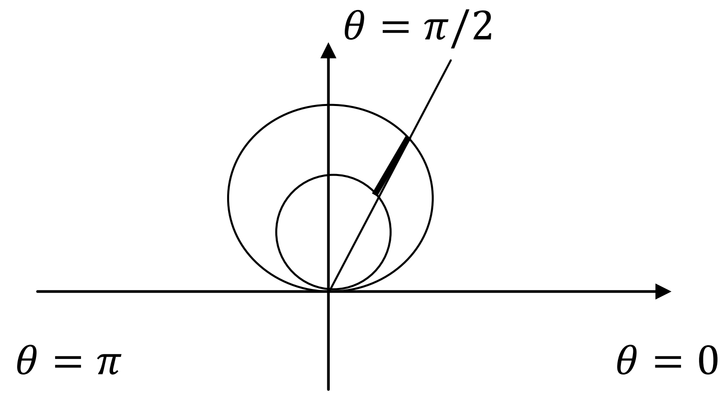
POLAR FORM

EXAMPLE :21

Find the area bounded by the circle

$r = 2\sin\theta \;\; and \;\; r = 4\sin\theta.$

Solution:



$$A = \int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^2}{2}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= 6\int_0^{\pi} (\sin\theta)^2 d\theta$$

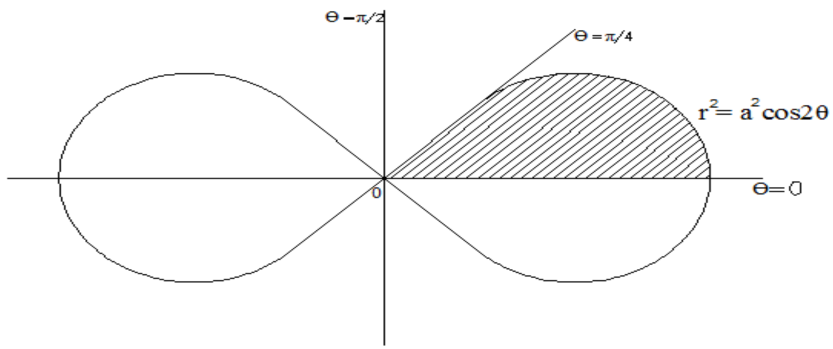
$$= 3 \int_0^{\pi} (1 - \cos 2\theta)\, d\theta$$

$$= 3\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 3\pi \; .$$

EXAMPLE :22

Find the area enclosed by the leminiscate $r^2 = a^2 \cos 2\theta$ by double integration.

Solution:



If r = 0 then $\cos 2\theta = 0$ implies $\theta = \frac{\pi}{4}$.

$$A = 4\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{a^2\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{\sqrt{a^2\cos 2\theta}} d\theta$$

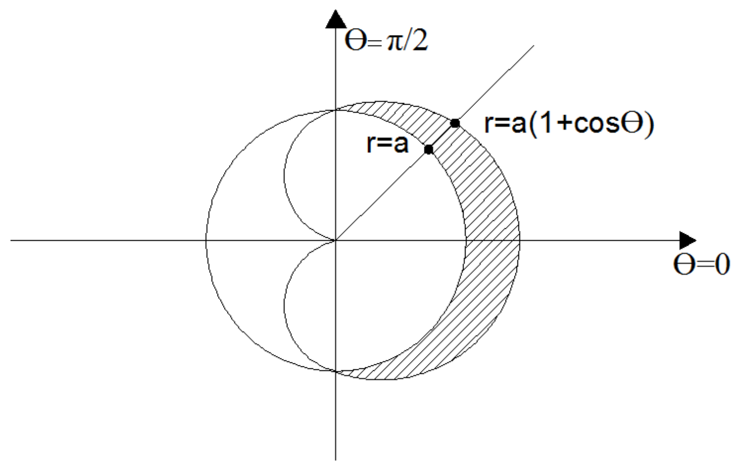
$$= 4a^2\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta}{2}\,d\theta$$

$$= 4\left[\frac{a^2\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}} = a^2.$$

EXAMPLE :23

Find the area that lies inside the cardioids $r = a(1 + \cos\theta)$ and outside the circle $r = a,$ by double integration.

Solution:



Solving $r = a(1 + \cos\theta)$ $and\ r = a$

$=> a(1 + \cos\theta) = a$

$=> \cos\theta = 0$

$=> \theta = \frac{\pi}{2}.$

$$A = 2\int_0^{\frac{\pi}{2}}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}}[(a(1+\cos\theta))^2 - a^2]d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}[2\cos\theta + (\cos\theta)^2]d\theta$$

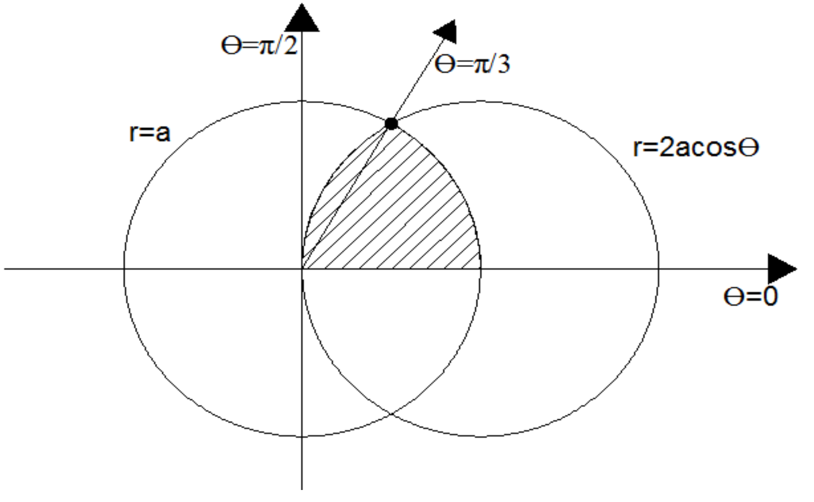
$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}[4\cos\theta + 1 + \cos 2\theta]\ d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2} + 4\sin\theta\right]_0^{\frac{\pi}{2}} = \frac{a^2}{2}(\pi + 8)\ .$$

EXAMPLE :24

Find the common area to the circles $r = a\,, r = 2a\cos\theta$.

Solution:



Given $r = a\,, r = 2a\cos\theta$, solving

⇨ $a = 2a\cos\theta$

⇨ $\cos\theta = \frac{1}{2}$

⇨ $\theta=\pi/3$

when $r = 0 => \cos\theta = 0 => \theta = \pi/2$

$$A = 2\iint r dr d\theta$$

$$= 2\int_0^{\frac{\pi}{3}}\int_0^{a} r\,dr\,d\theta + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2a\cos\,\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{3}}\left[\frac{r^2}{2}\right]_0^{a} d\theta + 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^{2a\cos\,\theta} d\theta$$

$$= a^2\int_0^{\frac{\pi}{3}} d\theta + 2a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(\cos\theta)^2 d\theta$$

$$= a^2[\theta]_0^{\frac{\pi}{3}} + 2a^2\left[\theta + \frac{\sin 2\,\theta}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= a^2\,\frac{\pi}{3} + 2a^2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) - a^2\,\frac{\sqrt{3}}{2}$$

$$= a^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

PROBLEMS FOR PRACTICE

1.Find by double integration, the area bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.

Ans: $\frac{16a^2}{3}\, sq.units.$

2.Find by double integration, the smallest area bounded by the circle $x^2 + y^2 = 9$ and the line $x + y = 3$.

Ans: $\frac{9}{4}\,(\pi - 2)sq.units.$

3.Find by double integration, the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2$.

Ans: $\left(\frac{1}{3} + \frac{\pi}{2}\right) sq\ units.$

4.Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the coordinate $r = a(1 - cos\theta)$.

Ans: $a^2\left(1 - \frac{\pi}{4}\right) sq.units.$