

Chapter 3: Multiple Integrals

Tutorial

Dr. D Bhanu Prakash

Unit 3: Tutorial

Author: Dr. D Bhanu Prakash

Vignan's Foundation for Science, Technology and Research

Vadlamudi - 522 213, Guntur, Andhra Pradesh, India.

Course: Calculus and Ordinary Differential Equations

Chapter 3: Multiple Integrals

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 4. **Applications: Computations of Areas and Volumes** - Basic geometric applications
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Problem Sets

1. Double Integrals

Problem 1.1: Evaluate the double integral $\int_0^2 \int_0^3 (x^2 + y) dy dx$.

Problem 1.2: Compute $\iint_R xy \, dA$ where R is the rectangle $[1, 3] \times [0, 2]$.

Problem 1.3: Evaluate $\int_0^1 \int_0^{1-x} (x + 2y) \, dy \, dx$ over the triangular region bounded by $x = 0$, $y = 0$, and $x + y = 1$.

Problem 1.4: Find $\iint_D e^{x+y} \, dA$ where D is the region bounded by $y = 0$, $y = x$, and $x = 1$.

Problem 1.5: The temperature distribution over a rectangular microchip ($0 \leq x \leq 4$ mm, $0 \leq y \leq 3$ mm) is given by $T(x, y) = 100 - 2x^2 - 3y^2$ degrees Celsius. Calculate the average temperature over the chip surface using $T_{avg} = \frac{1}{A} \iint_R T(x, y) \, dA$ where A is the area. This helps thermal engineers design cooling systems for integrated circuits.

2. Change of Order of Integration

Problem 2.1: Change the order of integration and evaluate: $\int_0^1 \int_y^1 \sin(x^2) \, dx \, dy$.

Problem 2.2: Reverse the order of integration: $\int_0^2 \int_{x^2}^4 f(x, y) \, dy \, dx$ and sketch both regions.

Problem 2.3: Evaluate $\int_0^1 \int_y^{\sqrt{y}} x^3 \, dx \, dy$ by first changing the order of integration.

Problem 2.4: Change the order of integration for $\int_0^4 \int_0^{\sqrt{x}} e^{y^3} \, dy \, dx$ and evaluate the resulting integral.

Problem 2.5: In image processing, blur detection involves integrating intensity gradients over triangular regions. Consider the integral $\int_0^{256} \int_x^{256} \frac{I(x, y)}{(x-y)^{1/3}} \, dy \, dx$ where $I(x, y)$ represents pixel intensity. The integrand has a singularity when integrated in this order. Change the order of integration to $\int_0^{256} \int_0^y \frac{I(x, y)}{(x-y)^{1/3}} \, dx \, dy$ to make numerical computation feasible. For simplicity, evaluate the case where $I(x, y) = 1$.

3. Triple Integrals

Problem 3.1: Evaluate $\int_0^1 \int_0^2 \int_0^3 xyz \, dz \, dy \, dx$.

Problem 3.2: Compute $\iiint_E (x + y + z) \, dV$ where E is the box $[0, 1] \times [0, 2] \times [0, 3]$.

Problem 3.3: Evaluate $\iiint_T z \, dV$ where T is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Problem 3.4: Find $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z) \, dz \, dy \, dx$.

Problem 3.5: A 3D-printed component has density varying with position according to $\rho(x, y, z) = 8 - x - y - z$ g/cm³ in the region bounded by $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 6$ (all

coordinates in cm). Calculate the total mass using $M = \iiint_V \rho(x, y, z) dV$. Variable density printing is used in aerospace to optimize strength-to-weight ratios.

4. Applications: Computations of Areas and Volumes

Problem 4.1: Find the area of the region enclosed by the cardioid $r = 1 + \cos \theta$ using $A = \frac{1}{2} \int_0^{2\pi} r^2 d\theta$.

Problem 4.2: Calculate the area of the region bounded by $y = x^2$ and $y = 2x$ using a double integral.

Problem 4.3: Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$ using a double integral.

Problem 4.4: Compute the volume of the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3 - y$.

Problem 4.5: A water tank design has a parabolic cross-section. The tank is formed by rotating the region between $z = x^2 + y^2$ and $z = 4$ around the z-axis, then intersecting with the cylinder $x^2 + y^2 = 3$. Calculate the volume of water the tank can hold using appropriate coordinates. Civil engineers need this for capacity calculations and structural load analysis. Express your integral in cylindrical coordinates.
