

Chapter 4: First Order Ordinary Differential Equations

Lecture

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Unit 4: Lecture Notes

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Course: Calculus and Ordinary Differential Equations

Chapter 4: First Order Ordinary Differential Equations

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Topic 21: Introduction — Order, Degree, and Formation of ODEs

21.1 What is a Differential Equation?

A **differential equation (DE)** is an equation that contains an unknown function and one or more of its derivatives.

Examples in Nature and Engineering: - The rate of cooling of a hot object depends on its temperature → leads to a DE - The current in a circuit depends on how fast the charge changes → leads to a DE - The position of a vibrating spring depends on its velocity and acceleration → leads to a DE

Ordinary Differential Equation (ODE): A DE involving derivatives with respect to **only one** independent variable.

$$\frac{dy}{dx} = 2x, \quad y'' + 3y' + 2y = 0, \quad x \frac{d^2y}{dx^2} + \sin x = 0$$

Partial Differential Equation (PDE): A DE involving partial derivatives with respect to **two or more** independent variables. (Not covered in this unit.)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{Laplace equation})$$

21.2 Order of a Differential Equation

Definition: The **order** of a differential equation is the order of the **highest derivative** present in the equation.

How to find order: Look for the highest derivative and count how many times it is differentiated.

Examples:

Equation	Highest Derivative	Order
$\frac{dy}{dx} = 3x^2 + 1$	$\frac{dy}{dx}$ (1st derivative)	1
$\frac{d^2y}{dx^2} + 4y = 0$	$\frac{d^2y}{dx^2}$ (2nd derivative)	2
$y''' - 2y'' + y = \sin x$	y''' (3rd derivative)	3
$\frac{d^4y}{dx^4} = x$	$\frac{d^4y}{dx^4}$ (4th derivative)	4
$\left(\frac{dy}{dx}\right)^3 + y = 0$	$\frac{dy}{dx}$ (1st derivative, though cubed)	1

Important Note: The **power** to which a derivative is raised does NOT affect the order — only the **level** of differentiation matters.

21.3 Degree of a Differential Equation

Definition: The **degree** of a differential equation is the **power (exponent)** of the highest order derivative, after the equation has been made **free from radicals and fractions** involving derivatives.

Step-by-step approach: 1. Clear all radicals and fractions involving derivatives 2. Identify the highest order derivative 3. Find its exponent (power) in the resulting polynomial form

Examples:

Example 21.3.1:

$$\left(\frac{d^2y}{dx^2}\right)^3 + 5\frac{dy}{dx} - y = 0$$

- Highest derivative: $\frac{d^2y}{dx^2}$ (order 2)
- Its power: 3
- **Order = 2, Degree = 3**

Example 21.3.2:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = 0$$

- Highest derivative: $\frac{d^2y}{dx^2}$ (order 2)
- Its power: 1 (even though lower derivative has power 4)
- **Order = 2, Degree = 1**

Example 21.3.3:

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$$

This has a radical — we must remove it.

Step 1: Square both sides:

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^2$$

- Highest derivative: $\frac{d^2y}{dx^2}$, raised to power 2
 - **Order = 2, Degree = 2**
-

Example 21.3.4:

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = k \frac{d^2y}{dx^2}$$

Step 1: Raise both sides to the power $\frac{2}{3}$... actually, **cube both sides:**

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right)^3 = k^2 \left(\frac{d^2y}{dx^2}\right)^2$$

- Highest derivative: $\frac{d^2y}{dx^2}$, raised to power 2
 - **Order = 2, Degree = 2**
-

Example 21.3.5:

$$\frac{d^2y}{dx^2} + \sin\left(\frac{dy}{dx}\right) = 0$$

- The term $\sin\left(\frac{dy}{dx}\right)$ cannot be expressed as a finite polynomial in $\frac{dy}{dx}$
 - **Degree is NOT DEFINED** (the equation is not a polynomial in derivatives)
-

21.4 Complete Classification Table

Equation	Order	Degree	Notes
$y' = x^2 + 1$	1	1	Linear

Equation	Order	Degree	Notes
$(y')^2 + y = 0$	1	2	Non-linear
$y'' - 5y' + 6y = 0$	2	1	Linear, homogeneous
$y'' + (y')^3 + y = e^x$	2	1	Non-linear (due to $(y')^3$)
$(y'')^2 + y' = x$	2	2	Non-linear
$y''' + y'' - y = \cos x$	3	1	Linear
$\sqrt{y''} + y = 0 \rightarrow$	2	1	After clearing radical
$y'' + y^2 = 0$			
$y' + e^y = 0$	1	Not defined	e^y is not polynomial in y'

21.5 Linear vs Non-Linear ODEs

A first-order ODE is **LINEAR** if it can be written as:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ depend only on x — **not** on y or its derivatives.

Non-linear = contains y^2 , $y \cdot y'$, $\sin(y)$, $(y')^2$, etc.

ODE	Linear?	Why
$y' + 2y = e^x$	Yes	$P = 2, Q = e^x$
$y' + y^2 = x$	No	y^2 term
$y' \cdot y = x$	No	Product of y' and y
$xy' - 3y = \sin x$	Yes	Rewrite as $y' - (3/x)y = (\sin x)/x$

21.6 Solution of a Differential Equation

General Solution: Contains as many arbitrary constants as the order of the ODE.

Particular Solution: Obtained by substituting specific initial or boundary conditions into the general solution to find the constants.

Singular Solution: A solution that cannot be obtained from the general solution by any choice of constant (rare).

Topic 22: Variable Separable Method

22.1 The Core Idea

If a first-order ODE can be written so that **all terms involving x (and dx) are on one side** and **all terms involving y (and dy) are on the other side**, we say the variables are **separable**.

Standard Form:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Separation Step:

$$\frac{dy}{g(y)} = f(x) dx$$

Integration:

$$\int \frac{dy}{g(y)} = \int f(x) dx + C$$

22.2 Step-by-Step Procedure

1. Rearrange the ODE into the form $\frac{dy}{dx} = f(x) \cdot g(y)$
 2. Rewrite as $\frac{dy}{g(y)} = f(x) dx$
 3. Integrate both sides
 4. Simplify (include $+C$ on one side only)
 5. Apply initial conditions if given
-

Example 22.1 — Basic Separation

Problem: Solve $\frac{dy}{dx} = \frac{x}{y}$.

Solution:

Step 1: Separate variables:

$$y \, dy = x \, dx$$

Step 2: Integrate both sides:

$$\int y \, dy = \int x \, dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

Step 3: Simplify:

$$y^2 = x^2 + C \quad \text{where } C = 2C_1$$

$$\boxed{y^2 - x^2 = C}$$

This is a family of **hyperbolas** (or the degenerate lines $y = \pm x$ when $C = 0$).

Example 22.2 — Trigonometric Separation

Problem: Solve $\frac{dy}{dx} = \sin x \cos^2 y$.

Solution:

Step 1: Separate:

$$\frac{dy}{\cos^2 y} = \sin x \, dx$$

$$\sec^2 y \, dy = \sin x \, dx$$

Step 2: Integrate:

$$\int \sec^2 y \, dy = \int \sin x \, dx$$

$$\tan y = -\cos x + C$$

$$\boxed{\tan y + \cos x = C}$$

Example 22.3 — Exponential Form with Initial Condition

Problem: Solve $\frac{dy}{dx} = e^{x+y}$, with $y(0) = 0$.

Solution:

Step 1: Rewrite:

$$\frac{dy}{dx} = e^x \cdot e^y$$

Step 2: Separate:

$$\frac{dy}{e^y} = e^x \, dx$$

$$e^{-y} \, dy = e^x \, dx$$

Step 3: Integrate:

$$\int e^{-y} \, dy = \int e^x \, dx$$

$$-e^{-y} = e^x + C$$

Step 4: Apply $y(0) = 0$:

$$-e^0 = e^0 + C \implies -1 = 1 + C \implies C = -2$$

Step 5: Particular solution:

$$-e^{-y} = e^x - 2$$

$$e^{-y} = 2 - e^x$$

$$\boxed{y = -\ln(2 - e^x)}$$

Domain note: Valid for $x < \ln 2$ since we need $2 - e^x > 0$.

Example 22.4 — Equation Separable after Rearranging

Problem: Solve $(1 + x^2) \frac{dy}{dx} = xy$.

Solution:

Step 1: Separate:

$$\frac{dy}{y} = \frac{x dx}{1 + x^2}$$

Step 2: Integrate (right side: let $u = 1 + x^2$, $du = 2x dx$):

$$\int \frac{dy}{y} = \int \frac{x dx}{1 + x^2}$$

$$\ln |y| = \frac{1}{2} \ln(1 + x^2) + \ln |C_1|$$

Step 3: Exponentiate:

$$|y| = C_1 \sqrt{1 + x^2}$$

$$\boxed{y = C \sqrt{1 + x^2}}$$

where C is an arbitrary constant (absorbing the \pm).

Example 22.5 — Separable with Partial Fractions

Problem: Solve $\frac{dy}{dx} = y(1 - y)$.

Solution:

Step 1: Separate:

$$\frac{dy}{y(1 - y)} = dx$$

Step 2: Use partial fractions on left:

$$\frac{1}{y(1 - y)} = \frac{1}{y} + \frac{1}{1 - y}$$

(Check: $\frac{1}{y} + \frac{1}{1 - y} = \frac{1 - y + y}{y(1 - y)} = \frac{1}{y(1 - y)}$ \square)

Step 3: Integrate:

$$\int \left(\frac{1}{y} + \frac{1}{1 - y} \right) dy = \int dx$$

$$\ln |y| - \ln |1 - y| = x + C_1$$

$$\ln \left| \frac{y}{1 - y} \right| = x + C_1$$

Step 4: Exponentiate:

$$\frac{y}{1-y} = e^{x+C_1} = Ae^x \quad (A = e^{C_1})$$

Step 5: Solve for y :

$$y = Ae^x(1-y) = Ae^x - Aye^x$$

$$y(1 + Ae^x) = Ae^x$$

$$y = \frac{Ae^x}{1 + Ae^x}$$

This is the **logistic function** — fundamental in population modelling and machine learning.

Topic 23: Homogeneous Equations and Reducible Forms

23.1 What is a Homogeneous ODE?

A first-order ODE $\frac{dy}{dx} = f(x, y)$ is **homogeneous** if $f(x, y)$ can be written as a function of $\frac{y}{x}$ alone — i.e., $f(x, y) = F\left(\frac{y}{x}\right)$.

Quick test: Replace x by λx and y by λy . If $f(\lambda x, \lambda y) = f(x, y)$ (the λ cancels completely), the equation is homogeneous.

Substitution: Let $y = vx$, so $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

The equation becomes **separable** in v and x .

23.2 Step-by-Step Procedure

1. Verify the equation is homogeneous
2. Substitute $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$
3. The equation reduces to variables separable in v and x
4. Separate and integrate

5. Back-substitute $v = \frac{y}{x}$

Example 23.1 — Standard Homogeneous Equation

Problem: Solve $\frac{dy}{dx} = \frac{x+y}{x}$.

Solution:

Step 1: Check homogeneity

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x} = F\left(\frac{y}{x}\right) \checkmark$$

Step 2: Substitute $y = vx$:

$$v + x \frac{dv}{dx} = 1 + v$$

Step 3: Simplify:

$$x \frac{dv}{dx} = 1$$

Step 4: Separate:

$$dv = \frac{dx}{x}$$

Step 5: Integrate:

$$v = \ln|x| + C$$

Step 6: Back-substitute $v = y/x$:

$$\frac{y}{x} = \ln|x| + C$$

$$\boxed{y = x \ln|x| + Cx}$$

Example 23.2 — Homogeneous with More Complexity**Problem:** Solve $(x^2 + y^2) dx - 2xy dy = 0$.**Solution:****Step 1:** Rewrite as:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

Check homogeneity: Replace $x \rightarrow \lambda x, y \rightarrow \lambda y$:

$$\frac{(\lambda x)^2 + (\lambda y)^2}{2(\lambda x)(\lambda y)} = \frac{\lambda^2(x^2 + y^2)}{2\lambda^2 xy} = \frac{x^2 + y^2}{2xy} \checkmark$$

Step 2: Substitute $y = vx, \frac{dy}{dx} = v + x \frac{dv}{dx}$:

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x(vx)} = \frac{1 + v^2}{2v}$$

Step 3: Simplify:

$$x \frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 + v^2 - 2v^2}{2v} = \frac{1 - v^2}{2v}$$

Step 4: Separate:

$$\frac{2v dv}{1 - v^2} = \frac{dx}{x}$$

Step 5: Integrate left side (let $u = 1 - v^2, du = -2v dv$):

$$\int \frac{2v dv}{1 - v^2} = -\ln |1 - v^2| = \ln \left| \frac{1}{1 - v^2} \right|$$

$$-\ln |1 - v^2| = \ln |x| + \ln |C_1|$$

$$\ln \frac{1}{|1 - v^2|} = \ln |C_1 x|$$

$$\frac{1}{1 - v^2} = Cx$$

Step 6: Back-substitute $v = y/x$:

$$\frac{1}{1 - y^2/x^2} = Cx$$

$$\frac{x^2}{x^2 - y^2} = Cx$$

$$\boxed{x^2 - y^2 = \frac{x}{C} = kx}$$

where $k = 1/C$ is the new arbitrary constant.

23.3 Equations Reducible to Homogeneous Form

Type: $\frac{dy}{dx} = \frac{ax + by + c}{px + qy + r}$ where $c \neq 0$ or $r \neq 0$.

Case 1: $\frac{a}{p} \neq \frac{b}{q}$ — lines intersect. Find the intersection (h, k) of $ax + by + c = 0$ and $px + qy + r = 0$. Substitute $x = X + h$, $y = Y + k$ to remove constants. The equation becomes homogeneous in X, Y .

Case 2: $\frac{a}{p} = \frac{b}{q}$ — lines are parallel. Let $z = ax + by$; the equation becomes separable.

Example 23.3 — Reducible Form (Case 2: Parallel Lines)

Problem: Solve $\frac{dy}{dx} = \frac{2x + 3y - 1}{4x + 6y - 5}$.

Solution:

Step 1: Check ratio: $\frac{a}{p} = \frac{2}{4} = \frac{1}{2}$ and $\frac{b}{q} = \frac{3}{6} = \frac{1}{2}$.

Since $\frac{a}{p} = \frac{b}{q}$, lines are parallel. Use substitution $z = 2x + 3y$.

Step 2: Differentiate $z = 2x + 3y$:

$$\frac{dz}{dx} = 2 + 3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3} \left(\frac{dz}{dx} - 2 \right)$$

Step 3: Rewrite the numerator and denominator in terms of z :

$$\text{Numerator: } 2x + 3y - 1 = z - 1$$

$$\text{Denominator: } 4x + 6y - 5 = 2(2x + 3y) - 5 = 2z - 5$$

Step 4: Substitute:

$$\frac{1}{3} \left(\frac{dz}{dx} - 2 \right) = \frac{z - 1}{2z - 5}$$

$$\frac{dz}{dx} - 2 = \frac{3(z - 1)}{2z - 5}$$

$$\frac{dz}{dx} = 2 + \frac{3(z - 1)}{2z - 5} = \frac{2(2z - 5) + 3z - 3}{2z - 5} = \frac{7z - 13}{2z - 5}$$

Step 5: Separate:

$$\frac{2z - 5}{7z - 13} dz = dx$$

Step 6: Partial fractions on left:

$$\frac{2z - 5}{7z - 13} = \frac{2}{7} + \frac{-5 + 26/7}{7z - 13} = \frac{2}{7} + \frac{-9/7}{7z - 13} = \frac{2}{7} - \frac{9}{7(7z - 13)}$$

Step 7: Integrate:

$$\int \left(\frac{2}{7} - \frac{9}{7(7z - 13)} \right) dz = \int dx$$

$$\frac{2z}{7} - \frac{9}{49} \ln |7z - 13| = x + C$$

Step 8: Back-substitute $z = 2x + 3y$:

$$\frac{2(2x + 3y)}{7} - \frac{9}{49} \ln |7(2x + 3y) - 13| = x + C$$

$$\frac{2(2x + 3y)}{7} - \frac{9}{49} \ln |14x + 21y - 13| = x + C$$

Topic 24: Linear First Order ODEs

24.1 Standard Form and Motivation

A **first-order linear ODE** has the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x only.

Why is this important? This is the most commonly occurring ODE in engineering applications — RC circuits, mixing problems, heat transfer, signal systems, and population models all reduce to this form.

Key property: Even when $Q(x) \neq 0$, this is **always solvable** by a systematic formula.

24.2 The Integrating Factor Method

Core Idea: Multiply both sides by a carefully chosen function $\mu(x)$ called the **integrating factor**, so the left side becomes an exact derivative $\frac{d}{dx}[\mu(x) \cdot y]$.

The integrating factor is:

$$\mu(x) = e^{\int P(x) dx}$$

Derivation:

Multiply $\frac{dy}{dx} + Py = Q$ by $\mu = e^{\int P dx}$:

$$e^{\int P dx} \frac{dy}{dx} + P e^{\int P dx} y = Q e^{\int P dx}$$

The left side is exactly $\frac{d}{dx} [e^{\int P dx} \cdot y]$ by the product rule!

$$\frac{d}{dx} [\mu y] = Q \mu$$

Integrate both sides:

$$\mu y = \int Q \mu dx + C$$

General Solution:

$$y = \frac{1}{\mu} \left[\int Q \mu dx + C \right]$$

24.3 Step-by-Step Procedure

1. Write in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$
 2. Identify $P(x)$ and $Q(x)$
 3. Compute $\mu = e^{\int P(x) dx}$ (no need for $+C$ here)
 4. Multiply both sides by μ
 5. Recognise LHS as $\frac{d}{dx} [\mu y]$
 6. Integrate both sides
 7. Solve for y
 8. Apply initial conditions if given
-

Example 24.1 — Basic Linear ODE

Problem: Solve $\frac{dy}{dx} + 2y = e^{3x}$.

Solution:

Step 1: Already in standard form. $P(x) = 2$, $Q(x) = e^{3x}$.

Step 2: Integrating factor:

$$\mu = e^{\int 2 dx} = e^{2x}$$

Step 3: Multiply both sides by e^{2x} :

$$e^{2x} \frac{dy}{dx} + 2e^{2x}y = e^{2x} \cdot e^{3x} = e^{5x}$$

Step 4: Recognise LHS:

$$\frac{d}{dx} [e^{2x}y] = e^{5x}$$

Step 5: Integrate both sides:

$$e^{2x}y = \int e^{5x} dx = \frac{e^{5x}}{5} + C$$

Step 6: Solve for y :

$$y = \frac{e^{3x}}{5} + Ce^{-2x}$$

Interpretation: As $x \rightarrow \infty$, the term $Ce^{-2x} \rightarrow 0$ (transient), and $y \rightarrow \frac{e^{3x}}{5}$ (the forced/steady response).

Example 24.2 — Variable Coefficient Linear ODE

Problem: Solve $\frac{dy}{dx} - \frac{y}{x} = x^2$.

Solution:

Step 1: Standard form. $P(x) = -\frac{1}{x}$, $Q(x) = x^2$.

Step 2: Integrating factor:

$$\mu = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = e^{\ln(x^{-1})} = \frac{1}{x}$$

Step 3: Multiply both sides by $\frac{1}{x}$:

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = x$$

Step 4: Recognise LHS:

$$\frac{d}{dx} \left[\frac{y}{x} \right] = x$$

Step 5: Integrate:

$$\frac{y}{x} = \int x dx = \frac{x^2}{2} + C$$

Step 6: Solve for y :

$$y = \frac{x^3}{2} + Cx$$

Verification: $y' = \frac{3x^2}{2} + C$. Then $y' - \frac{y}{x} = \frac{3x^2}{2} + C - \frac{x^2}{2} - C = x^2 \quad \square$

Example 24.3 — Linear ODE with Trigonometric Coefficient

Problem: Solve $\frac{dy}{dx} + y \tan x = \sec x$, $y(0) = 1$.

Solution:

Step 1: $P(x) = \tan x$, $Q(x) = \sec x$.

Step 2: Integrating factor:

$$\mu = e^{\int \tan x dx} = e^{\ln |\sec x|} = \sec x$$

Step 3: Multiply both sides by $\sec x$:

$$\sec x \frac{dy}{dx} + y \sec x \tan x = \sec^2 x$$

Step 4: Recognise LHS:

$$\frac{d}{dx}[y \sec x] = \sec^2 x$$

Step 5: Integrate:

$$y \sec x = \int \sec^2 x dx = \tan x + C$$

Step 6: Solve for y :

$$y = \sin x + C \cos x$$

Step 7: Apply $y(0) = 1$:

$$1 = \sin 0 + C \cos 0 = 0 + C \implies C = 1$$

$$\boxed{y = \sin x + \cos x}$$

Verification: $y' = \cos x - \sin x$. Then $y' + y \tan x = \cos x - \sin x + (\sin x + \cos x) \tan x$

$= \cos x - \sin x + \sin x + \cos x \tan x = \cos x + \cos x \cdot \frac{\sin x}{\cos x} = \cos x + \sin x \dots$ let's use another check:

$$\begin{aligned} y' + y \tan x &= (\cos x - \sin x) + (\sin x + \cos x) \frac{\sin x}{\cos x} \\ &= \cos x - \sin x + \frac{\sin^2 x + \sin x \cos x}{\cos x} \\ &= \frac{\cos^2 x - \sin x \cos x + \sin^2 x + \sin x \cos x}{\cos x} = \frac{1}{\cos x} = \sec x \quad \square \end{aligned}$$

Example 24.4 — Recognising Non-Standard Form

Problem: Solve $(x + 1)\frac{dy}{dx} + y = (x + 1)^3$.

Solution:

Step 1: Divide by $(x + 1)$ to get standard form:

$$\frac{dy}{dx} + \frac{1}{x + 1} y = (x + 1)^2$$

So $P(x) = \frac{1}{x + 1}$, $Q(x) = (x + 1)^2$.

Step 2: Integrating factor:

$$\mu = e^{\int \frac{1}{x+1} dx} = e^{\ln|x+1|} = x + 1$$

Step 3: Multiply both sides by $(x + 1)$:

$$\frac{d}{dx} [(x + 1)y] = (x + 1)^3$$

Step 4: Integrate:

$$(x + 1)y = \int (x + 1)^3 dx = \frac{(x + 1)^4}{4} + C$$

Step 5: Solve for y :

$$y = \frac{(x + 1)^3}{4} + \frac{C}{x + 1}$$

Example 24.5 — Linear ODE as x a function of y

Problem: Solve $\frac{dx}{dy} + \frac{2x}{y} = y^2$.

(Here x is the dependent variable and y is independent — this is valid!)

Solution:

Step 1: Standard form (in x). $P(y) = \frac{2}{y}$, $Q(y) = y^2$.

Step 2: Integrating factor:

$$\mu = e^{\int \frac{2}{y} dy} = e^{2 \ln y} = y^2$$

Step 3: Multiply both sides by y^2 :

$$\frac{d}{dy}[xy^2] = y^4$$

Step 4: Integrate:

$$xy^2 = \frac{y^5}{5} + C$$

Step 5: Solve for x :

$$x = \frac{y^3}{5} + \frac{C}{y^2}$$

Topic 25: Bernoulli's Equations (Further Reading)

25.1 What is a Bernoulli Equation?

A **Bernoulli equation** is a first-order ODE of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where n is any real number.

- If $n = 0$: reduces to standard linear ODE
- If $n = 1$: separable equation ($\frac{dy}{dx} = (Q - P)y$)
- If $n \neq 0, 1$: genuinely non-linear \rightarrow use Bernoulli's substitution

Historical note: This equation was first studied by Jacob Bernoulli in 1695 and was solved shortly after by Leibniz — the key insight being the substitution that linearises the equation.

25.4 Comparison of All Four Methods

Method	Form of ODE	Key Technique
Variable Separable	$\frac{dy}{dx} = f(x) \cdot g(y)$	Separate and integrate directly
Homogeneous	$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	Substitute $y = vx \rightarrow$ separable
Linear	$\frac{dy}{dx} + Py = Q$	Integrating factor $\mu = e^{\int P dx}$
Bernoulli	$\frac{dy}{dx} + Py = Qy^n$	Substitute $z = y^{1-n} \rightarrow$ linear

Topic 26: Applications — Natural Growth and Decay

26.1 The Core Mathematical Model

Many natural and technological processes follow a simple law: **the rate of change of a quantity is proportional to the quantity itself.**

$$\frac{dN}{dt} = kN$$

where: - $N(t)$ = quantity at time t - k = proportionality constant - $k > 0$: **growth** (population, compound interest, viral spread) - $k < 0$: **decay** (radioactive material, drug concentration, atmospheric pressure)

Solving the ODE:

This is separable:

$$\frac{dN}{N} = k dt$$

$$\int \frac{dN}{N} = \int k dt$$

$$\ln |N| = kt + C_1$$

$$N = e^{kt+C_1} = e^{C_1} \cdot e^{kt}$$

With initial condition $N(0) = N_0$:

$$N(t) = N_0 e^{kt}$$

This is the **universal law of exponential growth/decay**.

26.2 Key Derived Quantities

Doubling time T_d (growth): time for N to double.

$$N_0 e^{kT_d} = 2N_0 \implies kT_d = \ln 2 \implies T_d = \frac{\ln 2}{k}$$

Half-life $T_{1/2}$ (decay): time for N to halve. Here $k < 0$, let $k = -\lambda$:

$$N_0 e^{-\lambda T_{1/2}} = \frac{N_0}{2} \implies \lambda T_{1/2} = \ln 2 \implies T_{1/2} = \frac{\ln 2}{\lambda}$$

Carbon dating: The half-life of Carbon-14 is approximately **5730 years**. By measuring the fraction of C-14 remaining, archaeologists determine the age of organic material.

26.3 Natural Growth — Worked Examples

Example 26.1 — Bacterial Population Growth

Problem: A bacterial culture starts with 500 bacteria. After 3 hours, there are 2000 bacteria. (a) Find the growth constant k . (b) Find the number of bacteria after 8 hours. (c) When will there be 10,000 bacteria?

Real-life context: Understanding bacterial growth rates is critical in food safety (preventing contamination), biotechnology (fermenter design), and medical treatment (antibiotic dosing to outpace infection).

Solution:

Step 1: Set up the model

$$N(t) = N_0 e^{kt} = 500e^{kt}$$

Part (a): Find k

Use $N(3) = 2000$:

$$500e^{3k} = 2000$$

$$e^{3k} = 4$$

$$3k = \ln 4$$

$$k = \frac{\ln 4}{3} = \frac{2 \ln 2}{3} \approx \frac{1.386}{3} \approx 0.4621 \text{ hr}^{-1}$$

Part (b): Population after 8 hours

$$N(8) = 500e^{0.4621 \times 8} = 500e^{3.697}$$

$$e^{3.697} \approx 40.32$$

$$N(8) = 500 \times 40.32 \approx \mathbf{20,158} \text{ bacteria}$$

Exact form: $N(8) = 500 \cdot e^{(8 \ln 4)/3} = 500 \cdot 4^{8/3} = 500 \cdot 40.32 \approx 20,159$

Part (c): When does $N = 10,000$?

$$500e^{kt} = 10000$$

$$e^{kt} = 20$$

$$kt = \ln 20$$

$$t = \frac{\ln 20}{k} = \frac{\ln 20}{\ln 4/3} = \frac{3 \ln 20}{\ln 4} = \frac{3 \times 2.996}{1.386} \approx \frac{8.987}{1.386} \approx \mathbf{6.48 \text{ hours}}$$

Answers: - $k \approx 0.462 \text{ hr}^{-1}$ - After 8 hours: approximately **20,159 bacteria** - Reaches 10,000 after approximately **6.48 hours**

Example 26.2 — Radioactive Decay and Carbon Dating

Problem: A radioactive substance has a half-life of 8 days (similar to Iodine-131, used in thyroid treatment). (a) Find the decay constant λ . (b) What percentage remains after 30 days? (c) How many days until only 5% remains?

Real-life context: Radioactive isotopes are used in cancer treatment (radiation therapy), medical imaging (PET scans), and nuclear power. Knowing the decay rate determines safe dosage, waste storage duration, and equipment scheduling.

Solution:

Step 1: Model

$$N(t) = N_0 e^{-\lambda t}$$

Part (a): Find λ

Half-life $T_{1/2} = 8$ days:

$$N_0 e^{-8\lambda} = \frac{N_0}{2}$$

$$e^{-8\lambda} = \frac{1}{2}$$

$$-8\lambda = -\ln 2$$

$$\lambda = \frac{\ln 2}{8} \approx \frac{0.6931}{8} \approx 0.0866 \text{ day}^{-1}$$

Part (b): Percentage remaining after 30 days

$$\frac{N(30)}{N_0} = e^{-\lambda \times 30} = e^{-30 \times 0.0866} = e^{-2.598}$$

$$e^{-2.598} \approx 0.0743$$

Approximately 7.43% remains after 30 days.

Alternatively, using half-life formula:

$$\frac{N(30)}{N_0} = \left(\frac{1}{2}\right)^{30/8} = (0.5)^{3.75} \approx 0.0743$$

Part (c): When does 5% remain?

$$e^{-\lambda t} = 0.05$$

$$-\lambda t = \ln(0.05)$$

$$t = \frac{-\ln(0.05)}{\lambda} = \frac{\ln 20}{0.0866} = \frac{2.996}{0.0866} \approx \mathbf{34.6 \text{ days}}$$

Answers: - $\lambda = \frac{\ln 2}{8} \approx 0.0866 \text{ day}^{-1}$ - After 30 days: **7.43% remains** - 5% remains after **34.6 days**

Nuclear medicine note: A patient treated with Iodine-131 needs to avoid close contact with others for a safe period. After ~35 days (≈ 4 half-lives), only 5% of the original dose remains active.

26.4 Key Insight: The Power of the Exponential Model

The equation $N(t) = N_0 e^{kt}$ appears across virtually every field:

Application	$N(t)$ represents	$k > 0$ or $k < 0$?
Population biology	Number of organisms	$k > 0$ (growth)
Radioactive decay	Number of atoms	$k < 0$ (decay)
Compound interest	Account balance	$k > 0$ (growth)
Drug metabolism	Drug concentration	$k < 0$ (decay)
Atmospheric pressure	Air pressure at height h	$k < 0$ (decreases with altitude)
Capacitor discharge	Voltage across capacitor	$k < 0$ (decay)
Moore's Law	Number of transistors	$k > 0$ (approximately)

Topic 27: Applications — Newton's Law of Cooling and Other Models

27.1 Newton's Law of Cooling — Statement

Newton's Law of Cooling states that the rate of change of temperature of a body is proportional to the **difference between the body's temperature and the ambient (surrounding) temperature**.

$$\frac{dT}{dt} = -k(T - T_s)$$

where: - $T = T(t)$ = temperature of the body at time t - T_s = temperature of the surrounding medium (constant) - $k > 0$ = positive cooling constant - The negative sign ensures cooling: if $T > T_s$, then $\frac{dT}{dt} < 0$ (body cools)

Note: The same equation with $k > 0$ also applies to **heating** — if $T < T_s$, the body heats up.

27.2 Solving the Governing ODE

Step 1: Separate variables:

$$\frac{dT}{T - T_s} = -k dt$$

Step 2: Integrate both sides:

$$\ln |T - T_s| = -kt + C_1$$

Step 3: Exponentiate:

$$T - T_s = Ae^{-kt} \quad (A = e^{C_1})$$

Step 4: Apply initial condition $T(0) = T_0$:

$$T_0 - T_s = A$$

$$T(t) = T_s + (T_0 - T_s)e^{-kt}$$

Physical interpretation: - At $t = 0$: $T = T_0$ \square - As $t \rightarrow \infty$: $T \rightarrow T_s$ \square (body eventually reaches room temperature) - The temperature difference $T - T_s$ decays exponentially

27.3 Finding the Cooling Constant

Given two data points (t_1, T_1) and (t_2, T_2) :

$$T_1 - T_s = (T_0 - T_s)e^{-kt_1}$$

$$T_2 - T_s = (T_0 - T_s)e^{-kt_2}$$

Dividing:

$$\frac{T_2 - T_s}{T_1 - T_s} = e^{-k(t_2 - t_1)}$$

$$k = \frac{1}{t_2 - t_1} \ln \frac{T_1 - T_s}{T_2 - T_s}$$

Example 27.1 — Classic Cooling Problem

Problem: A metal rod is heated to 200°C . It is placed in a room where the temperature is 25°C . After 10 minutes, the rod cools to 120°C .

- (a) Find the cooling constant k .
- (b) What is the temperature after 30 minutes?
- (c) When will the temperature reach 40°C ?

Real-life context: Cooling problems appear in **manufacturing** (steel quenching — cooling a hot metal in water or oil to give it desired hardness and microstructure), **food safety** (ensuring cooked food cools fast enough to prevent bacterial growth), and **electronics** (how quickly a processor cools after a heavy compute task).

Solution:

Given: $T_0 = 200^{\circ}\text{C}$, $T_s = 25^{\circ}\text{C}$, $T(10) = 120^{\circ}\text{C}$.

The model:

$$T(t) = 25 + (200 - 25)e^{-kt} = 25 + 175e^{-kt}$$

Part (a): Find k

$$T(10) = 25 + 175e^{-10k} = 120$$

$$175e^{-10k} = 95$$

$$e^{-10k} = \frac{95}{175} = \frac{19}{35}$$

$$-10k = \ln\left(\frac{19}{35}\right) = \ln 19 - \ln 35$$

$$k = -\frac{1}{10} \ln\left(\frac{19}{35}\right) = \frac{1}{10} \ln\left(\frac{35}{19}\right)$$

$$k = \frac{\ln(35/19)}{10} = \frac{\ln(1.842)}{10} \approx \frac{0.6116}{10} \approx 0.0612 \text{ min}^{-1}$$

Part (b): Temperature after 30 minutes

$$T(30) = 25 + 175e^{-0.0612 \times 30} = 25 + 175e^{-1.836}$$

$$e^{-1.836} \approx 0.1593$$

$$T(30) = 25 + 175 \times 0.1593 = 25 + 27.88 \approx \mathbf{52.9^\circ C}$$

$$\text{Exact form: } T(30) = 25 + 175 \left(\frac{19}{35}\right)^3 = 25 + 175 \times \frac{6859}{42875} = 25 + 28.0 = 53.0^\circ C$$

Part (c): When does temperature reach 40°C?

$$25 + 175e^{-kt} = 40$$

$$175e^{-kt} = 15$$

$$e^{-kt} = \frac{15}{175} = \frac{3}{35}$$

$$-kt = \ln\left(\frac{3}{35}\right)$$

$$t = \frac{\ln(35/3)}{k} = \frac{\ln(11.667)}{0.0612} = \frac{2.457}{0.0612} \approx \mathbf{40.1 \text{ minutes}}$$

Answers: - $k \approx 0.0612 \text{ min}^{-1}$ - After 30 minutes: $T \approx \mathbf{52.9^\circ C}$ - Reaches $40^\circ C$ at approximately $t \approx \mathbf{40.1 \text{ minutes}}$

Example 27.2 — Finding Initial Temperature

Problem: A bowl of soup is placed in a room at 20°C . After 5 minutes it cools from an unknown initial temperature to 80°C , and after 10 minutes it is at 65°C . Find:

- (a) The initial temperature of the soup.
- (b) The cooling constant k .

Real-life context: This type of two-point measurement is used in **forensic science** to estimate the time of death — the body temperature is measured at two times after discovery, and Newton's Law of Cooling is applied to determine when the body was at 37°C (normal living temperature). It is also used in food service to ensure proper hot-holding temperatures.

Solution:

Model: $T(t) = 20 + (T_0 - 20)e^{-kt}$

Given: $T(5) = 80, T(10) = 65$.

Step 1: Write the two equations

$$T(5) = 20 + (T_0 - 20)e^{-5k} = 80 \quad \Rightarrow \quad (T_0 - 20)e^{-5k} = 60 \quad \dots (1)$$

$$T(10) = 20 + (T_0 - 20)e^{-10k} = 65 \quad \Rightarrow \quad (T_0 - 20)e^{-10k} = 45 \quad \dots (2)$$

Step 2: Divide equation (2) by equation (1)

$$\frac{(T_0 - 20)e^{-10k}}{(T_0 - 20)e^{-5k}} = \frac{45}{60}$$

$$e^{-5k} = \frac{3}{4}$$

$$-5k = \ln\left(\frac{3}{4}\right)$$

$$k = \frac{\ln(4/3)}{5} = \frac{\ln(1.333)}{5} \approx \frac{0.2877}{5} \approx 0.0575 \text{ min}^{-1}$$

Step 3: Find T_0 from equation (1)

$$T_0 - 20 = \frac{60}{e^{-5k}} = 60 \cdot e^{5k} = \frac{60}{3/4} = 60 \times \frac{4}{3} = 80$$

$$T_0 = 80 + 20 = \mathbf{100^\circ C}$$

Answers: - **Initial temperature:** $T_0 = 100^\circ C$ (the soup was at boiling point!) - **Cooling constant:** $k \approx 0.0575 \text{ min}^{-1}$

Verification: $T(5) = 20 + 80e^{-0.0575 \times 5} = 20 + 80(0.75) = 20 + 60 = 80^\circ C \quad \square$

$T(10) = 20 + 80e^{-0.0575 \times 10} = 20 + 80(0.75)^2 = 20 + 80(0.5625) = 20 + 45 = 65^\circ C \quad \square$

27.4 Mixing Problems — Another ODE Application

Concept: A tank contains a solution. A different concentration of solution flows in; the mixture flows out. We find how the amount of substance in the tank changes over time.

Rate of change = Rate In – Rate Out

$$\frac{dS}{dt} = (\text{concentration in}) \times (\text{flow rate in}) - (\text{concentration out}) \times (\text{flow rate out})$$

The **concentration out** = $\frac{S(t)}{\text{Volume}(t)}$ since the mixture is assumed uniform.

Example 27.3 — Mixing / Dilution Problem

Problem: A tank contains 200 litres of brine (salt dissolved in water) with 50 kg of dissolved salt. Pure water is pumped in at 5 L/min, and the well-mixed solution flows out at 5 L/min.

- Set up the ODE for $S(t)$, the amount of salt (kg) at time t (min).
- Solve the ODE.
- How long until only 10 kg of salt remains?

Real-life context: This models **water treatment plants** (gradual dilution of contaminants), **blood dialysis** (removing waste from blood by dilution), and **chemical reactor design** (maintaining desired concentrations).

Solution:

Step 1: Identify rates

- Volume is constant at 200 L (in-rate = out-rate = 5 L/min)
- Flow in: pure water, so salt rate in = $0 \text{ kg/L} \times 5 \text{ L/min} = 0$
- Flow out: concentration = $\frac{S(t)}{200} \text{ kg/L}$, so salt rate out = $\frac{S(t)}{200} \times 5 = \frac{S}{40} \text{ kg/min}$

Step 2: ODE

$$\frac{dS}{dt} = 0 - \frac{S}{40} = -\frac{S}{40}$$

Step 3: This is a separable (exponential decay) ODE

$$\frac{dS}{S} = -\frac{dt}{40}$$

$$\ln |S| = -\frac{t}{40} + C_1$$

$$S(t) = Ae^{-t/40}$$

Step 4: Apply initial condition $S(0) = 50 \text{ kg}$

$$A = 50$$

$$S(t) = 50e^{-t/40} \text{ kg}$$

Part (c): When does $S = 10 \text{ kg}$?

$$50e^{-t/40} = 10$$

$$e^{-t/40} = \frac{1}{5}$$

$$-\frac{t}{40} = -\ln 5$$

$$t = 40 \ln 5 \approx 40 \times 1.609 \approx \mathbf{64.4} \text{ minutes}$$

Interpretation: After about 64 minutes (just over 3 tank volumes of water), the salt has been diluted to 20% of its original amount.

27.5 Summary of All Application Models

Model	ODE	Solution	Parameters
Exponential Growth	$\dot{N} = kN, k > 0$	$N = N_0 e^{kt}$	Doubling time $T_d = \frac{\ln 2}{k}$
Exponential Decay	$\dot{N} = -\lambda N$	$N = N_0 e^{-\lambda t}$	Half-life $T_{1/2} = \frac{\ln 2}{\lambda}$
Newton's Cooling	$\dot{T} = -k(T - T_s)$	$T = T_s + (T_0 - T_s)e^{-kt}$	Approaches T_s as $t \rightarrow \infty$
Mixing/Dilution	$\dot{S} = \text{rate in} - \text{rate out}$	Depends on setup	Volume may be variable

Overall Summary of Unit 4 (Topics 21–27)

Topic	Core Concept	Key Method/Formula
21. Intro: Order, Degree, Formation	Classify ODEs; derive ODE from family of curves	Eliminate arbitrary constants by differentiation
22. Variable Separable	$\frac{dy}{dx} = f(x)g(y)$	Separate: $\frac{dy}{g(y)} = f(x) dx$; integrate
23. Homogeneous + Reducible	$\frac{dy}{dx} = F(y/x)$	Substitute $y = vx$; reduces to separable
24. Linear First Order	$\frac{dy}{dx} + Py = Q$	Integrating factor $\mu = e^{\int P dx}$

Topic	Core Concept	Key Method/Formula
25. Bernoulli's Equation	$\frac{dy}{dx} + Py = Qy^n$	Substitute $z = y^{1-n}$; reduces to linear
26. Growth and Decay	$\frac{dN}{dt} = kN$	$N(t) = N_0 e^{kt}$; half-life, doubling time
27. Newton's Cooling + Mixing	$\frac{dT}{dt} = -k(T - T_s)$	$T(t) = T_s + (T_0 - T_s)e^{-kt}$; mixing: rate in - rate out
