

Chapter 4: First Order Ordinary Differential Equations

Tutorial

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Unit 4: Tutorial

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Course: Calculus and Ordinary Differential Equations

Chapter 4: First Order Ordinary Differential Equations

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Problem Sets

1. Introduction: Order, Degree, and Formation of ODEs

Problem 1.1: Determine the order and degree of the differential equation: $(y'')^3 + 2(y')^2 - 3y = \sin x$.

Problem 1.2: Find the order and degree of: $\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2} = k\frac{d^2y}{dx^2}$.

2. Variable Separable Method

Problem 2.1: Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y}$.

Problem 2.2: Find the general solution of $\frac{dy}{dx} = e^{x-y}$.

Problem 2.3: Solve $(1 + x^2)\frac{dy}{dx} = xy$ with the initial condition $y(0) = 2$.

Problem 2.4: Solve $y' = \frac{1+y^2}{1+x^2}$ and find the particular solution passing through the origin.

Problem 2.5: In a chemical reaction, the rate of formation of a product is proportional to the product of concentrations of two reactants. If $x(t)$ is the amount of product at time t , and the initial amounts of reactants are a and b , the differential equation is:

$$\frac{dx}{dt} = k(a - x)(b - x)$$

where k is the rate constant. Solve this equation by separation of variables for the case where $a = 5$ mol, $b = 3$ mol, and $x(0) = 0$. This model is fundamental in reaction kinetics for pharmaceutical synthesis.

3. Homogeneous Equations and Reducible Forms

Problem 3.1: Solve the homogeneous equation $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Problem 3.2: Solve $(x^2 + y^2)dx - 2xy dy = 0$ using the substitution $y = vx$.

Problem 3.3: Solve $\frac{dy}{dx} = \frac{x+2y}{2x+y}$ by recognizing it as homogeneous.

Problem 3.4: Solve the equation $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$ by first reducing it to homogeneous form using an appropriate substitution.

Problem 3.5: In population dynamics with competition, two species with populations $x(t)$ and

$y(t)$ can be related through:

$$\frac{dy}{dx} = \frac{y(2 - x - y)}{x(1 - x - y)}$$

This is a homogeneous equation when considering the ratio of populations. Solve this equation to find the relationship between x and y in equilibrium states. Such models help ecologists predict outcomes of species competition in ecosystems and are used in conservation planning.

4. Linear First Order ODEs

Problem 4.1: Solve the linear differential equation $\frac{dy}{dx} + 2y = e^{3x}$.

Problem 4.2: Find the general solution of $\frac{dy}{dx} - \frac{y}{x} = x^2$.

Problem 4.3: Solve $\frac{dy}{dx} + y \tan x = \sec x$ with the initial condition $y(0) = 1$.

Problem 4.4: Solve $(x + 1)\frac{dy}{dx} + y = (x + 1)^3$.

Problem 4.5: An RL circuit (resistor-inductor in series) connected to a voltage source $E(t) = E_0 \sin(\omega t)$ satisfies the differential equation:

$$L \frac{dI}{dt} + RI = E_0 \sin(\omega t)$$

where I is current, L is inductance, and R is resistance. Solve this linear ODE for $I(t)$ with $L = 0.5$ H, $R = 10 \Omega$, $E_0 = 100$ V, $\omega = 60\pi$ rad/s, and $I(0) = 0$. This determines the current response in AC circuits, crucial for power supply design in electronic devices.

5. Applications: Natural Growth and Decay

Problem 5.1: A bacterial culture grows at a rate proportional to its size. If the population doubles in 3 hours and initially there are 1000 bacteria, find the population after 10 hours.

Problem 5.2: A radioactive substance decays at a rate proportional to the amount present. If 20% of the substance decays in 5 years, find the half-life of the substance.

Problem 5.3: The population of a city grows exponentially from 100,000 to 150,000 in 10 years. What will be the population after 25 years assuming the same growth rate?

Problem 5.4: A sample contains 100 mg of a radioactive isotope with half-life 8 days. How much remains after 30 days? When will only 10 mg remain?

Problem 5.5: Moore's Law suggests that the number of transistors on integrated circuits doubles approximately every 2 years. A microprocessor in 2020 had $N_0 = 10 \times 10^9$ transistors.

- (a) Write and solve the differential equation $\frac{dN}{dt} = kN$ to find $N(t)$.
- (b) Predict the number of transistors in 2030.
- (c) At what year would we expect 100 billion transistors?

This exponential growth model drives semiconductor industry roadmaps and helps tech companies plan manufacturing capabilities and product releases.

6. Applications: Newton's Law of Cooling and Other Models

Problem 6.1: A body at temperature 80°C is placed in a room at 20°C . After 10 minutes, its temperature is 60°C . Find the temperature after 20 minutes using Newton's Law of Cooling: $\frac{dT}{dt} = -k(T - T_s)$ where T_s is the surrounding temperature.

Problem 6.2: A cup of coffee at 90°C is left in a room at 25°C . If it cools to 70°C in 5 minutes, when will it reach 40°C ?

Problem 6.3: A thermometer reading 18°C is brought into a room where the temperature is 25°C . After 2 minutes, the thermometer reads 22°C . Find the thermometer reading after 5 minutes.

Problem 6.4: A tank contains 100 liters of brine with 20 kg of dissolved salt. Pure water enters at 5 L/min and the mixture (kept uniform) leaves at 5 L/min. Set up and solve the differential equation for the amount of salt $S(t)$ at time t .

Problem 6.5: A smartphone processor operates at 70°C during intensive computation. When the phone is put to sleep, the processor cools according to Newton's Law of Cooling with ambient temperature 25°C . Temperature sensors show the processor reaches 50°C after 30 seconds of sleep mode.

- (a) Find the cooling constant k .
- (b) Determine when the processor reaches 30°C (safe temperature for pocket storage).
- (c) If thermal throttling activates at 65°C during use, how long after waking from 30°C can the processor run at full load if heat generation adds 2°C per second?

Set up and solve: $\frac{dT}{dt} = -k(T - 25)$ for cooling and $\frac{dT}{dt} = 2 - k(T - 25)$ for active use.

This thermal management analysis is critical for mobile device design, affecting battery life and user safety.