Chapter 5: Second Order Ordinary Differential Equations

Lecture

Dr. D Bhanu Prakash

Unit 5: Lecture Notes

Author: Dr. D Bhanu Prakash

Vignan's Foundation for Science, Technology and Research
Vadlamudi - 522 213, Guntur, Andhra Pradesh, India.

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Topic 28: Introduction — Homogeneous and Non-Homogeneous Equations

28.1 What is a Second Order ODE?

A **second order ordinary differential equation** is an equation involving an unknown function $y(x)$, its first derivative y', and its second derivative y''.

General Form:

$$F(x, y, y', y'') = 0$$

or in **standard form** (solved for y''):

$$y'' + P(x)\,y' + Q(x)\,y = R(x)$$

where $P(x)$, $Q(x)$, and $R(x)$ are functions of x (or constants).

28.2 Classification

Homogeneous Second Order ODE:

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

The right-hand side is **zero**. The equation has no external forcing term.

Non-Homogeneous Second Order ODE:

$$y'' + P(x)\,y' + Q(x)\,y = R(x), \quad R(x) \neq 0$$

The right-hand side $R(x)$ is called the **forcing function**, **input**, or **non-homogeneous term**.

28.3 The Superposition Principle

Theorem: If $y_1(x)$ and $y_2(x)$ are both solutions of the homogeneous equation:

$$y'' + P(x)y' + Q(x)y = 0$$

then $c_1 y_1 + c_2 y_2$ is also a solution for any constants c_1, c_2.

28.4 Linear Independence and the Wronskian

Two functions y_1 and y_2 are **linearly independent** if neither is a constant multiple of the other.

The **Wronskian** of y_1 and y_2 is:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

- If $W \neq 0$ on an interval, y_1 and y_2 are linearly independent on that interval.
- If $W = 0$, they are linearly dependent.

28.5 Structure of the General Solution

For the **non-homogeneous** equation $y'' + Py' + Qy = R(x)$, the **general solution** is:

$$\boxed{y = y_c + y_p}$$

where: - y_c = **Complementary Function (CF)** = general solution of the associated homogeneous equation $y'' + Py' + Qy = 0$ - y_p = **Particular Integral (PI)** = any one particular solution of the full non-homogeneous equation

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28.6 Identifying Homogeneous vs Non-Homogeneous

Example set:

Equation	Type	Reason
$y'' - 5y' + 6y = 0$	Homogeneous	RHS $= 0$
$y'' + 4y' + 4y = e^x$	Non-Homogeneous	RHS $= e^x \neq 0$
$y'' - 3y' = 0$	Homogeneous	RHS $= 0$
$y'' + y = \sin x$	Non-Homogeneous	RHS $= \sin x \neq 0$
$y'' + 2y' + y = x^2 + 1$	Non-Homogeneous	RHS $= x^2 + 1 \neq 0$

Topic 29: Complementary Functions (CF) — Homogeneous Solutions with Constant Coefficients

29.1 Setting Up the Characteristic Equation

For the homogeneous ODE with **constant coefficients**:

$$ay'' + by' + cy = 0 \quad (a \neq 0)$$

We try the solution $y = e^{mx}$. Then:

$$y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

Substituting:

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

Since $e^{mx} \neq 0$, we need:

$$\boxed{am^2 + bm + c = 0}$$

This is the **Characteristic Equation** (also called Auxiliary Equation).

29.2 Three Cases Based on Discriminant $\Delta = b^2 - 4ac$

The nature of roots m_1, m_2 of the characteristic equation determines the form of CF:

Case	Roots	Discriminant	CF Form
Case 1	Two distinct real roots $m_1 \neq m_2$	$\Delta > 0$	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case	Roots	Discriminant	CF Form
Case 2	Two equal (repeated) real roots $m_1 = m_2 = m$	$\Delta = 0$	$y_c = (c_1 + c_2 x)e^{mx}$
Case 3	Two complex conjugate roots $m = \alpha \pm i\beta$	$\Delta < 0$	$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

Case 1: Distinct Real Roots — Three Worked Examples

Example 29.1.1

Problem: Find the complementary function of $y'' - 5y' + 6y = 0$.

Solution:

Step 1: Write the characteristic equation.

$$m^2 - 5m + 6 = 0$$

Step 2: Factorise.

$$(m - 2)(m - 3) = 0$$

$$m_1 = 2, \quad m_2 = 3$$

Step 3: Since $m_1 \neq m_2$ (distinct real roots), the CF is:

$$\boxed{y_c = c_1 e^{2x} + c_2 e^{3x}}$$

Verification: Differentiate y_c and substitute back:

$y_c' = 2c_1 e^{2x} + 3c_2 e^{3x}$

$y_c'' = 4c_1 e^{2x} + 9c_2 e^{3x}$

$y_c'' - 5y_c' + 6y_c = (4c_1 e^{2x} + 9c_2 e^{3x}) - 5(2c_1 e^{2x} + 3c_2 e^{3x}) + 6(c_1 e^{2x} + c_2 e^{3x})$

$= c_1 e^{2x}(4 - 10 + 6) + c_2 e^{3x}(9 - 15 + 6) = 0 + 0 = 0$ □

Example 29.1.2

Problem: Solve $y'' - y' - 12y = 0$.

Solution:

Step 1: Characteristic equation:

$$m^2 - m - 12 = 0$$

Step 2: Factorise:

$$(m - 4)(m + 3) = 0$$

$$m_1 = 4, \quad m_2 = -3$$

Step 3: Distinct real roots, so:

$$\boxed{y = c_1 e^{4x} + c_2 e^{-3x}}$$

Example 29.1.3

Problem: Solve $2y'' + 3y' - 2y = 0$.

Solution:

Step 1: Characteristic equation (divide by 2 or keep as is):

$$2m^2 + 3m - 2 = 0$$

Step 2: Use quadratic formula:

$$m = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4}$$

$$m_1 = \frac{-3+5}{4} = \frac{2}{4} = \frac{1}{2}, \quad m_2 = \frac{-3-5}{4} = \frac{-8}{4} = -2$$

Step 3:

$$\boxed{y = c_1 e^{x/2} + c_2 e^{-2x}}$$

Case 2: Repeated Real Roots — Three Worked Examples

Example 29.2.1

Problem: Find the CF of $y'' - 6y' + 9y = 0$.

Solution:

Step 1: Characteristic equation:

$$m^2 - 6m + 9 = 0$$

Step 2: Factorise:

$$(m-3)^2 = 0$$

$$m_1 = m_2 = 3 \quad \text{(repeated root)}$$

Step 3: For repeated roots $m = 3$, the CF is:

$$\boxed{y_c = (c_1 + c_2 x)e^{3x}}$$

Why this form? If we used $y = c_1 e^{3x} + c_2 e^{3x} = (c_1 + c_2)e^{3x}$, this would only be one independent solution. The factor x is introduced to generate a second linearly independent solution.

Verification:

$y = (c_1 + c_2 x)e^{3x}$

$y' = c_2 e^{3x} + 3(c_1 + c_2 x)e^{3x} = e^{3x}(3c_1 + c_2 + 3c_2 x)$

$y'' = e^{3x}\left[3(3c_1 + c_2 + 3c_2 x) + 3c_2\right] = e^{3x}(9c_1 + 6c_2 + 9c_2 x)$

$y'' - 6y' + 9y = e^{3x}[(9c_1 + 6c_2 + 9c_2 x) - 6(3c_1 + c_2 + 3c_2 x) + 9(c_1 + c_2 x)]$

$= e^{3x}[9c_1 + 6c_2 + 9c_2 x - 18c_1 - 6c_2 - 18c_2 x + 9c_1 + 9c_2 x]$

$= e^{3x}[(9 - 18 + 9)c_1 + (6 - 6)c_2 + (9 - 18 + 9)c_2 x] = 0$ □

Example 29.2.2

Problem: Solve $y'' + 4y' + 4y = 0$.

Solution:

Step 1: Characteristic equation:

$$m^2 + 4m + 4 = 0$$

Step 2: Factorise:

$$(m+2)^2 = 0$$

$$m = -2 \quad \text{(repeated)}$$

Step 3:

$$\boxed{y = (c_1 + c_2 x)e^{-2x}}$$

Example 29.2.3

Problem: Solve $4y'' - 4y' + y = 0$.

Solution:

Step 1: Characteristic equation:

$$4m^2 - 4m + 1 = 0$$

Step 2: Factorise:

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2} \quad \text{(repeated)}$$

Step 3:

$$\boxed{y = (c_1 + c_2 x)e^{x/2}}$$

Case 3: Complex Conjugate Roots — Three Worked Examples

Example 29.3.1

Problem: Solve $y'' - 2y' + 5y = 0$.

Solution:

Step 1: Characteristic equation:

$$m^2 - 2m + 5 = 0$$

Step 2: Use the quadratic formula:

$$m = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

So $\alpha = 1$, $\beta = 2$.

Step 3: Complex roots → use exponential-trigonometric form:

$$\boxed{y_c = e^{x}(c_1 \cos 2x + c_2 \sin 2x)}$$

Physical meaning: This solution represents **damped oscillations** — the e^x factor describes growth (since $\alpha = 1 > 0$) while the trigonometric part gives oscillation at frequency $\beta = 2$.

Example 29.3.2

Problem: Solve $y'' + 6y' + 13y = 0$.

Solution:

Step 1: Characteristic equation:

$$m^2 + 6m + 13 = 0$$

Step 2: Quadratic formula:

$$m = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

So $\alpha = -3$, $\beta = 2$.

Step 3:

$$\boxed{y_c = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x)}$$

Physical meaning: This represents **damped decaying oscillations** since $\alpha = -3 < 0$.

Example 29.3.3

Problem: Solve $y'' + 9y = 0$ with initial conditions $y(0) = 3$, $y'(0) = -6$.

Solution:

Step 1: Characteristic equation:

$$m^2 + 9 = 0 \implies m^2 = -9 \implies m = \pm 3i$$

Here $\alpha = 0$, $\beta = 3$.

Step 2: General solution:

$$y = e^{0 \cdot x}(c_1 \cos 3x + c_2 \sin 3x) = c_1 \cos 3x + c_2 \sin 3x$$

Step 3: Apply initial conditions.

At $x = 0$: $y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 3$

So $c_1 = 3$.

$y' = -3c_1 \sin 3x + 3c_2 \cos 3x$

At $x = 0$: $y'(0) = -3c_1(0) + 3c_2(1) = 3c_2 = -6$

So $c_2 = -2$.

Step 4: Particular solution:

$$\boxed{y = 3\cos 3x - 2\sin 3x}$$

This represents **pure undamped oscillation** (no decay or growth since $\alpha = 0$).

29.3 Summary of CF Determination

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Write ODE → Form Characteristic Equation am² + bm + c = 0
                          |
              Compute Δ = b² - 4ac
                /         |          \
           Δ > 0       Δ = 0       Δ < 0
         (Distinct)   (Repeated)  (Complex)
        m1 != m2 real  m1 = m2 = m  m = $\alpha$ ± i$\beta$
             |              |            |
  c1e^(m1x) + c2e^(m2x)  (c1+c2x)e^(mx)  e^($\alpha$x)[c1cos$\beta$x + c2sin$\beta$x]
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Topic 30: Particular Integrals — Method of Undetermined Coefficients

30.1 Introduction

The **Method of Undetermined Coefficients** is used to find a particular integral (PI) y_p for the non-homogeneous equation:

$$ay'' + by' + cy = R(x)$$

Core idea: We guess the form of y_p based on the form of $R(x)$, then substitute back into the ODE to find the unknown coefficients.

This method works when $R(x)$ **is one of:** - A polynomial: $R(x) = x^n, x^2 + 3x, $ etc. - An exponential: $R(x) = e^{kx}$ - Sine or cosine: $R(x) = \sin \beta x$ or $\cos \beta x$ - Products of the above

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30.2 Trial Solution Table

Form of $R(x)$	Trial solution y_p
k (constant)	A
kx^n	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$
ke^{ax}	Ae^{ax}
$k \sin(\beta x)$ or $k \cos(\beta x)$	$A \cos(\beta x) + B \sin(\beta x)$
$ke^{ax} \sin(\beta x)$ or $ke^{ax} \cos(\beta x)$	$e^{ax}(A \cos \beta x + B \sin \beta x)$
$kx^n e^{ax}$	$(A_n x^n + \cdots + A_0)e^{ax}$

IMPORTANT MODIFICATION RULE: If the trial solution y_p happens to be a solution of the homogeneous equation (i.e., appears in the CF), **multiply the trial solution by** x. If this also appears in CF, multiply by x^2.

30.3 Case A: Exponential Forcing Function $R(x) = ke^{ax}$

Example 30.A.1

Problem: Solve $y'' + 3y' + 2y = 6e^{3x}$.

Step 1: Find the CF

Characteristic equation: $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0 \implies m_1 = -1,\ m_2 = -2$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2: Trial solution for PI

$R(x) = 6e^{3x}$. Note: $a = 3$ is NOT a root of the characteristic equation, so no modification needed.

Trial: $y_p = Ae^{3x}$

Step 3: Find derivatives

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

Step 4: Substitute into ODE

$$9Ae^{3x} + 3(3Ae^{3x}) + 2(Ae^{3x}) = 6e^{3x}$$

$$e^{3x}(9A + 9A + 2A) = 6e^{3x}$$

$$20A = 6 \implies A = \frac{3}{10}$$

Step 5: Write PI

$$y_p = \frac{3}{10}e^{3x}$$

Step 6: General Solution

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \frac{3}{10}e^{3x}}$$

Example 30.A.2

Problem: Solve $y'' - 3y' + 2y = e^{2x}$.

Step 1: Find the CF

Characteristic equation: $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0 \implies m_1 = 1,\ m_2 = 2$

$$y_c = c_1 e^x + c_2 e^{2x}$$

Step 2: Check for modification

$R(x) = e^{2x}$ has $a = 2$. But $m = 2$ IS a root of the characteristic equation!

So the normal trial Ae^{2x} is part of CF — we must modify.

Modified trial: $y_p = Axe^{2x}$

Step 3: Derivatives

$y_p' = Ae^{2x} + 2Axe^{2x} = Ae^{2x}(1 + 2x)$

$y_p'' = 2Ae^{2x}(1 + 2x) + 2Ae^{2x} = Ae^{2x}(4x + 4) = 4Ae^{2x}(x + 1)$

Step 4: Substitute

$$4Ae^{2x}(x+1) - 3Ae^{2x}(1+2x) + 2Axe^{2x} = e^{2x}$$

$$Ae^{2x}[4x + 4 - 3 - 6x + 2x] = e^{2x}$$

$$Ae^{2x}[0 \cdot x + 1] = e^{2x}$$

$$A \cdot 1 = 1 \implies A = 1$$

Step 5:

$$y_p = xe^{2x}$$

Step 6: General Solution

$$\boxed{y = c_1 e^x + c_2 e^{2x} + x e^{2x}}$$

Example 30.A.3

Problem: Solve $y'' - 4y' + 4y = 3e^{2x}$.

Step 1: Find the CF

Characteristic equation: $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \implies m = 2$ (repeated root)

$$y_c = (c_1 + c_2 x)e^{2x}$$

Step 2: Check modification

$R(x) = 3e^{2x}$, $a = 2$.

- Normal trial Ae^{2x} is in CF (since e^{2x} is a solution).
- Modified trial Axe^{2x} is also in CF (since xe^{2x} is a solution).
- So we must use: $y_p = Ax^2 e^{2x}$

Step 3: Derivatives

$y_p = Ax^2 e^{2x}$

$y_p' = 2Axe^{2x} + 2Ax^2 e^{2x} = 2Axe^{2x}(1 + x)$

$y_p'' = 2Ae^{2x}(1 + x) + 2Axe^{2x} \cdot 2(1 + x) + 2Axe^{2x}$

Let's compute carefully:

$y_p' = 2Axe^{2x} + 2Ax^2 e^{2x}$

$y_p'' = 2Ae^{2x} + 4Axe^{2x} + 4Axe^{2x} + 4Ax^2 e^{2x} = e^{2x}(2A + 8Ax + 4Ax^2)$

Step 4: Substitute

$$e^{2x}(2A + 8Ax + 4Ax^2) - 4e^{2x}(2Ax + 2Ax^2) + 4Ax^2 e^{2x} = 3e^{2x}$$

$$e^{2x}[2A + 8Ax + 4Ax^2 - 8Ax - 8Ax^2 + 4Ax^2] = 3e^{2x}$$

$$e^{2x}[2A + (8A - 8A)x + (4A - 8A + 4A)x^2] = 3e^{2x}$$

$$2A = 3 \implies A = \frac{3}{2}$$

Step 5: General Solution

$$\boxed{y = (c_1 + c_2 x)e^{2x} + \frac{3}{2}x^2 e^{2x}}$$

30.4 Case B: Trigonometric Forcing Function $R(x) = k \sin \beta x$ or $k \cos \beta x$

Example 30.B.1

Problem: Solve $y'' + 4y' + 3y = \sin 2x$.

Step 1: CF

$m^2 + 4m + 3 = 0 \implies (m + 1)(m + 3) = 0 \implies m = -1, -3$

$$y_c = c_1 e^{-x} + c_2 e^{-3x}$$

Step 2: Trial PI

Since $R(x) = \sin 2x$, try:

$$y_p = A \cos 2x + B \sin 2x$$

Note: $\pm 2i$ are NOT roots of characteristic equation, so no modification needed.

Step 3: Derivatives

$y_p' = -2A \sin 2x + 2B \cos 2x$

$y_p'' = -4A\cos 2x - 4B\sin 2x$

Step 4: Substitute

$$(-4A\cos 2x - 4B\sin 2x) + 4(-2A\sin 2x + 2B\cos 2x) + 3(A\cos 2x + B\sin 2x) = \sin 2x$$

Collect $\cos 2x$ **terms:**

$$-4A + 8B + 3A = 0 \implies -A + 8B = 0 \implies A = 8B \quad \cdots (1)$$

Collect $\sin 2x$ **terms:**

$$-4B - 8A + 3B = 1 \implies -B - 8A = 1 \quad \cdots (2)$$

Solve (1) and (2):

Substitute $A = 8B$ into (2):

$$-B - 8(8B) = 1 \implies -B - 64B = 1 \implies -65B = 1 \implies B = -\frac{1}{65}$$

$$A = 8B = -\frac{8}{65}$$

Step 5: PI

$$y_p = -\frac{8}{65}\cos 2x - \frac{1}{65}\sin 2x$$

Step 6: General Solution

$$\boxed{y = c_1 e^{-x} + c_2 e^{-3x} - \frac{8}{65}\cos 2x - \frac{1}{65}\sin 2x}$$

Example 30.B.2

Problem: Solve $y'' + 4y = 8\cos 2x$.

Step 1: CF

$m^2 + 4 = 0 \implies m = \pm 2i \implies \alpha = 0, \beta = 2$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

Step 2: Check modification

Normal trial would be $A \cos 2x + B \sin 2x$. But $\cos 2x$ and $\sin 2x$ already appear in CF!

Modified trial: $y_p = x(A \cos 2x + B \sin 2x)$

Step 3: Derivatives

$y_p = Ax \cos 2x + Bx \sin 2x$

$y_p' = A \cos 2x - 2Ax \sin 2x + B \sin 2x + 2Bx \cos 2x$

$y_p'' = -2A \sin 2x - 2A \sin 2x - 4Ax \cos 2x + 2B \cos 2x + 2B \cos 2x - 4Bx \sin 2x$

$y_p'' = -4A \sin 2x - 4Ax \cos 2x + 4B \cos 2x - 4Bx \sin 2x$

Step 4: Substitute into $y'' + 4y = 8 \cos 2x$

$$(-4A \sin 2x - 4Ax \cos 2x + 4B \cos 2x - 4Bx \sin 2x) + 4(Ax \cos 2x + Bx \sin 2x) = 8 \cos 2x$$

$$-4A \sin 2x + 4B \cos 2x + x(-4A + 4A) \cos 2x + x(-4B + 4B) \sin 2x = 8 \cos 2x$$

$$-4A \sin 2x + 4B \cos 2x = 8 \cos 2x$$

Comparing coefficients:

$\cos 2x$: $4B = 8 \implies B = 2$

$\sin 2x$: $-4A = 0 \implies A = 0$

Step 5:

$$y_p = 2x \sin 2x$$

Step 6: General Solution

$$\boxed{y = c_1 \cos 2x + c_2 \sin 2x + 2x \sin 2x}$$

Example 30.B.3

Problem: Solve $y'' - y' + y = 2\cos x + 3\sin x$.

Step 1: CF

$m^2 - m + 1 = 0$

$$m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_c = e^{x/2}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$$

Step 2: Trial PI

Since $\pm i$ are NOT roots of characteristic equation:

$$y_p = A\cos x + B\sin x$$

Step 3: Derivatives

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

Step 4: Substitute

$$(-A\cos x - B\sin x) - (-A\sin x + B\cos x) + (A\cos x + B\sin x) = 2\cos x + 3\sin x$$

$$\cos x(-A - B + A) + \sin x(-B + A + B) = 2\cos x + 3\sin x$$

$$-B\cos x + A\sin x = 2\cos x + 3\sin x$$

Comparing:

$\cos x$: $-B = 2 \implies B = -2$

$\sin x$: $A = 3$

Step 5: General Solution

$$\boxed{y = e^{x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right) + 3\cos x - 2\sin x}$$

30.5 Case C: Polynomial Forcing Function $R(x) = P_n(x)$

Example 30.C.1

Problem: Solve $y'' + 3y' + 2y = 4x^2$.

Step 1: CF

$m^2 + 3m + 2 = 0 \implies (m+1)(m+2) = 0 \implies m = -1, -2$

$$y_c = c_1e^{-x} + c_2e^{-2x}$$

Step 2: Trial PI

$R(x) = 4x^2$ is a degree-2 polynomial, so try:

$$y_p = Ax^2 + Bx + C$$

Step 3: Derivatives

$y_p' = 2Ax + B$

$y_p'' = 2A$

Step 4: Substitute

$$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$$

$$2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = 4x^2$$

Comparing coefficients:

x^2: $2A = 4 \implies A = 2$

x^1: $6A + 2B = 0 \implies 12 + 2B = 0 \implies B = -6$

x^0: $2A + 3B + 2C = 0 \implies 4 - 18 + 2C = 0 \implies C = 7$

Step 5: General Solution

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + 2x^2 - 6x + 7}$$

Example 30.C.2

Problem: Solve $y'' - 2y' = 6x$.

Step 1: CF

$m^2 - 2m = 0 \implies m(m-2) = 0 \implies m = 0, 2$

$$y_c = c_1 + c_2 e^{2x}$$

Step 2: Check modification

Normal trial for $R(x) = 6x$ would be $Ax + B$.

But B is a constant, and $m = 0$ (which gives a constant in CF). Since the constant term c_1 is in CF, multiply by x:

Modified trial: $y_p = x(Ax + B) = Ax^2 + Bx$

Step 3: Derivatives

$y_p' = 2Ax + B$

$y_p'' = 2A$

Step 4: Substitute

$$2A - 2(2Ax + B) = 6x$$

$$2A - 2B - 4Ax = 6x$$

Comparing:

x: $-4A = 6 \implies A = -\frac{3}{2}$

x^0: $2A - 2B = 0 \implies -3 - 2B = 0 \implies B = -\frac{3}{2}$

Step 5: General Solution

$$\boxed{y = c_1 + c_2 e^{2x} - \frac{3}{2}x^2 - \frac{3}{2}x}$$

Topic 31: Particular Integrals — Variation of Parameters - Further Reading

31.1 Introduction

The Method of Variation of Parameters is a **general method** that works for any forcing function $R(x)$ — including functions like $\tan x$, $\ln x$, $\sec x$, or $\frac{1}{x}$, which cannot be handled by undetermined coefficients.

Given: $y'' + P(x)y' + Q(x)y = R(x)$

Assumption: The CF is $y_c = c_1 y_1 + c_2 y_2$ where y_1, y_2 are known.

Key Idea: Replace the constants c_1, c_2 with functions $u_1(x), u_2(x)$:

$$y_p = u_1(x)\, y_1 + u_2(x)\, y_2$$

31.2 The Formula

The functions u_1 and u_2 are found by solving:

$$u_1' = -\frac{y_2 \cdot R(x)}{W(y_1, y_2)}, \qquad u_2' = \frac{y_1 \cdot R(x)}{W(y_1, y_2)}$$

where the **Wronskian** is:

$$W = W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

Then integrate:

$$u_1 = \int -\frac{y_2 R}{W}\, dx, \qquad u_2 = \int \frac{y_1 R}{W}\, dx$$

And the PI is:

$$y_p = u_1 y_1 + u_2 y_2$$

Note: Write the equation in standard form (coefficient of y'' must be 1) before applying this formula.

Topic 32: Complete Solutions and Initial Value Problems (IVPs)

32.1 The Complete Solution

Recall the structure:

$$y_{general} = y_c + y_p$$

where: - $y_c = c_1 y_1 + c_2 y_2$ is the **complementary function** (CF) with two arbitrary constants c_1, c_2 - y_p is the **particular integral** (PI) — a specific function with no arbitrary constants

The **complete solution** (also called the **general solution**) is obtained by combining CF and PI.

32.2 Initial Value Problems (IVPs)

An **Initial Value Problem (IVP)** consists of a second-order ODE plus two conditions specified at the same point $x = x_0$:

$$y(x_0) = y_0, \quad y'(x_0) = y_1$$

Procedure: 1. Find the CF → includes c_1, c_2 2. Find the PI 3. Write complete solution: $y = y_c + y_p$ 4. Differentiate: $y' = y_c' + y_p'$ 5. Apply the two initial conditions to get two equations in c_1, c_2 6. Solve for c_1 and c_2 7. Substitute back to get the unique particular solution

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Example 32.1 — Exponential Forcing with IVP

Problem: Solve the IVP:

$$y'' - 3y' + 2y = 4e^x, \quad y(0) = 1,\ y'(0) = 0$$

Step 1: Find the CF

Characteristic equation: $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0 \implies m_1 = 1,\ m_2 = 2$$

$$y_c = c_1 e^x + c_2 e^{2x}$$

Step 2: Find the PI

$R(x) = 4e^x$. Note: $a = 1$ is a root ($m_1 = 1$), so modify.

Trial: $y_p = Axe^x$

$y_p' = Ae^x + Axe^x = Ae^x(1+x)$

$y_p'' = Ae^x + Ae^x(1+x) = Ae^x(2+x)$

Substitute into ODE:

$$Ae^x(2+x) - 3Ae^x(1+x) + 2Axe^x = 4e^x$$

$$Ae^x[(2+x) - (3+3x) + 2x] = 4e^x$$

$$Ae^x[2+x-3-3x+2x] = 4e^x$$

$$Ae^x[-1+0\cdot x] = 4e^x$$

$$-A = 4 \implies A = -4$$

$$y_p = -4xe^x$$

Step 3: Complete Solution

$$y = c_1e^x + c_2e^{2x} - 4xe^x$$

Step 4: Derivative

$$y' = c_1e^x + 2c_2e^{2x} - 4e^x - 4xe^x$$

$$y' = (c_1-4)e^x + 2c_2e^{2x} - 4xe^x$$

Step 5: Apply initial conditions

At $x = 0$:

$$y(0) = c_1e^0 + c_2e^0 - 4(0)e^0 = c_1 + c_2 = 1 \quad \cdots(1)$$

$$y'(0) = (c_1 - 4)e^0 + 2c_2e^0 - 0 = c_1 - 4 + 2c_2 = 0 \quad \cdots (2)$$

Step 6: Solve for constants

From (2): $c_1 + 2c_2 = 4 \quad \cdots (2')$

Subtract (1) from (2'): $c_2 = 3$

Substitute in (1): $c_1 + 3 = 1 \implies c_1 = -2$

Step 7: Particular solution of IVP

$$\boxed{y = -2e^x + 3e^{2x} - 4xe^x}$$

Verification: At $x = 0$: $y = -2 + 3 - 0 = 1$ □

$y' = -2e^x + 6e^{2x} - 4e^x - 4xe^x$; at $x = 0$: $y' = -2 + 6 - 4 - 0 = 0$ □

Example 32.2 — Polynomial Forcing with IVP

Problem: Solve the IVP:

$$y'' + 4y' + 4y = 8x^2 + 4x, \quad y(0) = -1,\ y'(0) = 3$$

Step 1: CF

Characteristic equation: $m^2 + 4m + 4 = 0$

$(m + 2)^2 = 0 \implies m = -2$ (repeated)

$$y_c = (c_1 + c_2x)e^{-2x}$$

Step 2: PI

$R(x) = 8x^2 + 4x$ (degree 2 polynomial). Trial: $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

Substitute:

$$2A + 4(2Ax + B) + 4(Ax^2 + Bx + C) = 8x^2 + 4x$$

$$4Ax^2 + (8A + 4B)x + (2A + 4B + 4C) = 8x^2 + 4x + 0$$

Comparing coefficients:

x^2: $4A = 8 \implies A = 2$

x^1: $8A + 4B = 4 \implies 16 + 4B = 4 \implies B = -3$

x^0: $2A + 4B + 4C = 0 \implies 4 - 12 + 4C = 0 \implies C = 2$

$$y_p = 2x^2 - 3x + 2$$

Step 3: Complete Solution

$$y = (c_1 + c_2 x)e^{-2x} + 2x^2 - 3x + 2$$

Step 4: Derivative

$$y' = c_2 e^{-2x} - 2(c_1 + c_2 x)e^{-2x} + 4x - 3$$

$$y' = (-2c_1 + c_2 - 2c_2 x)e^{-2x} + 4x - 3$$

Step 5: Apply initial conditions

At $x = 0$:

$$y(0) = c_1 e^0 + 0 + 0 - 0 + 2 = c_1 + 2 = -1 \implies c_1 = -3 \quad \cdots (1)$$

$$y'(0) = (-2c_1 + c_2)e^0 - 3 = -2c_1 + c_2 - 3 = 3 \quad \cdots (2)$$

From (2): $-2(-3) + c_2 = 6 \implies 6 + c_2 = 6 \implies c_2 = 0$

Step 6: Final Solution

With $c_1 = -3$, $c_2 = 0$:

$$\boxed{y = -3e^{-2x} + 2x^2 - 3x + 2}$$

Verification:

At $x = 0$: $y = -3 + 0 - 0 + 2 = -1$ □

$y' = 6e^{-2x} + 4x - 3$; at $x = 0$: $y' = 6 - 3 = 3$ □

Example 32.3 — Trigonometric Forcing with IVP

Problem: Solve the IVP:

$$y'' + 9y = 18 \sin x, \quad y(0) = 2, \ y'(0) = 3$$

Step 1: CF

Characteristic equation: $m^2 + 9 = 0 \implies m = \pm 3i$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

Step 2: PI

$R(x) = 18\sin x$. $\beta = 1$, and $\pm i$ are NOT roots of char. equation ($m = \pm 3i$), so no modification.

Trial: $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

Substitute:

$$(-A\cos x - B\sin x) + 9(A\cos x + B\sin x) = 18\sin x$$

$$8A\cos x + 8B\sin x = 18\sin x$$

Comparing:

$\cos x$: $8A = 0 \implies A = 0$

$\sin x$: $8B = 18 \implies B = \frac{9}{4}$

$$y_p = \frac{9}{4}\sin x$$

Step 3: Complete Solution

$$y = c_1\cos 3x + c_2\sin 3x + \frac{9}{4}\sin x$$

Step 4: Derivative

$$y' = -3c_1\sin 3x + 3c_2\cos 3x + \frac{9}{4}\cos x$$

Step 5: Apply initial conditions

At $x = 0$:

$$y(0) = c_1(1) + c_2(0) + \frac{9}{4}(0) = c_1 = 2 \implies c_1 = 2 \quad \cdots(1)$$

$$y'(0) = -3c_1(0) + 3c_2(1) + \frac{9}{4}(1) = 3c_2 + \frac{9}{4} = 3$$

$$3c_2 = 3 - \frac{9}{4} = \frac{12-9}{4} = \frac{3}{4} \implies c_2 = \frac{1}{4}$$

Step 6: Final Solution

$$\boxed{y = 2\cos 3x + \frac{1}{4}\sin 3x + \frac{9}{4}\sin x}$$

Verification:

At $x = 0$: $y = 2(1) + 0 + 0 = 2$ □

$y'(0) = 0 + \frac{3}{4} + \frac{9}{4} = \frac{12}{4} = 3$ □

Example 32.4 — Complete Solution using Variation of Parameters with IVP

Problem: Solve the IVP:

$$y'' - y = 2x, \quad y(0) = 0,\ y'(0) = 0$$

Step 1: CF

$m^2 - 1 = 0 \implies m = \pm 1$

$$y_c = c_1 e^x + c_2 e^{-x}$$

Step 2: PI by Undetermined Coefficients

$R(x) = 2x$. Trial: $y_p = Ax + B$

$y_p' = A, y_p'' = 0$

Substitute: $0 - (Ax + B) = 2x$

$-A = 2 \implies A = -2; -B = 0 \implies B = 0$

$$y_p = -2x$$

Step 3: Complete Solution

$$y = c_1 e^x + c_2 e^{-x} - 2x$$

Step 4: Derivative

$$y' = c_1 e^x - c_2 e^{-x} - 2$$

Step 5: Apply ICs

At $x = 0$:

$$y(0) = c_1 + c_2 = 0 \quad \cdots (1)$$

$$y'(0) = c_1 - c_2 - 2 = 0 \implies c_1 - c_2 = 2 \quad \cdots (2)$$

Adding (1) and (2): $2c_1 = 2 \implies c_1 = 1$

From (1): $c_2 = -1$

Step 6: Final Solution

$$\boxed{y = e^x - e^{-x} - 2x}$$

Note: $e^x - e^{-x} = 2\sinh x$, so this can also be written as $y = 2\sinh x - 2x$.

Example 32.5 — Complex Roots, Combined Forcing and IVP

Problem: Solve the IVP:

$$y'' + 2y' + 5y = 4e^{-x}\cos 2x, \quad y(0) = 1,\ y'(0) = -1$$

Step 1: CF

$m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$y_c = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

Step 2: PI

$R(x) = 4e^{-x}\cos 2x$.

The "base" of the exponential is -1 and the trig frequency is 2, so we check whether $-1 \pm 2i$ are roots. They ARE the characteristic roots!

Modified trial: Since $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ appear in CF, multiply by x:

$$y_p = xe^{-x}(A\cos 2x + B\sin 2x)$$

Differentiate: Let $u = e^{-x}(A\cos 2x + B\sin 2x)$ so $y_p = xu$.

$u = e^{-x}(A\cos 2x + B\sin 2x)$

$u' = e^{-x}[(-A + 2B)\cos 2x + (-2A - B)\sin 2x]$

$y_p' = u + xu'$

$y_p'' = 2u' + xu''$

$u'' = e^{-x}[(A - 4B)\cos 2x + (4A + B - 4A)\sin 2x]$

Wait, let's compute u'' carefully:

$u = e^{-x}(A\cos 2x + B\sin 2x)$

$u' = -e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}(-2A\sin 2x + 2B\cos 2x)$

$u' = e^{-x}[(-A+2B)\cos 2x + (-B-2A)\sin 2x]$

$u'' = e^{-x}[(A-2B-2(-A+2B))\cos 2x + (B+2A-2(-B-2A))\sin 2x]$

$= e^{-x}[(A-2B+2A-4B)\cos 2x + (B+2A+2B+4A)\sin 2x]$

$= e^{-x}[(3A-6B)\cos 2x + (6A+3B)\sin 2x]$

$= 3e^{-x}[(A-2B)\cos 2x + (2A+B)\sin 2x]$

Substitute $y_p = xu$, $y_p' = u + xu'$, $y_p'' = 2u' + xu''$ into the ODE:

$y_p'' + 2y_p' + 5y_p = (2u' + xu'') + 2(u + xu') + 5xu$

$= 2u' + 2u + x(u'' + 2u' + 5u)$

But u satisfies the **homogeneous** equation (since $e^{-x}\cos 2x$ is in CF):

$u'' + 2u' + 5u = 0$

Therefore:

$$y_p'' + 2y_p' + 5y_p = 2u' + 2u$$

We need $2u' + 2u = 4e^{-x}\cos 2x$, i.e., $u' + u = 2e^{-x}\cos 2x$.

$u' + u = e^{-x}[(-A+2B)\cos 2x + (-B-2A)\sin 2x] + e^{-x}[A\cos 2x + B\sin 2x]$

$= e^{-x}[(2B)\cos 2x + (-2A)\sin 2x]$

Set equal to $2e^{-x}\cos 2x$:

$\cos 2x$: $2B = 2 \implies B = 1$

$\sin 2x$: $-2A = 0 \implies A = 0$

$$y_p = xe^{-x}(0\cdot\cos 2x + 1\cdot\sin 2x) = xe^{-x}\sin 2x$$

Step 3: Complete Solution

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + xe^{-x} \sin 2x$$

Step 4: Derivative

$$y' = -e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-x}(-2c_1 \sin 2x + 2c_2 \cos 2x) + e^{-x} \sin 2x + xe^{-x}(2 \cos 2x - \sin 2x)$$

At $x = 0$:

$$y'(0) = -(c_1) + (2c_2) + 0 + 0 = -c_1 + 2c_2$$

Step 5: Apply ICs

At $x = 0$:

$$y(0) = c_1 = 1 \quad \cdots (1)$$

$$y'(0) = -c_1 + 2c_2 = -1 \implies -1 + 2c_2 = -1 \implies c_2 = 0 \quad \cdots (2)$$

Step 6: Final Solution

$$\boxed{y = e^{-x} \cos 2x + xe^{-x} \sin 2x}$$

This can be written compactly as:

$$y = e^{-x}(\cos 2x + x \sin 2x)$$

Physical Interpretation: This represents a **resonant response** — the $xe^{-x} \sin 2x$ term grows linearly with x before the exponential decay dominates. This is analogous to resonance in a damped mechanical or electrical system.

32.3 Summary: Step-by-Step Procedure for Solving IVPs

```
STEP 1: Find CF
    → Write characteristic equation
    → Find roots m1, m2
    → Write y_c (3 cases)

STEP 2: Find PI
    → Check form of R(x)
    → Choose method: Undetermined Coefficients or Variation of Parameters
    → Check modification rule
    → Solve for constants A, B, etc.
    → Write y_p

STEP 3: Complete Solution
    → y = y_c + y_p

STEP 4: Differentiate
    → Compute y'

STEP 5: Apply Initial Conditions
    → Substitute x = x0 in y: get equation for c1, c2
    → Substitute x = x0 in y': get second equation
    → Solve the system of 2 equations

STEP 6: Write Final Answer
    → Substitute c1, c2 back
    → Always verify by substitution
```

32.4 Quick Summary Table for Complete Solutions

Component	Formula	Role
CF: Distinct roots $m_1 \neq m_2$	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$	Transient (general homogeneous behaviour)
CF: Repeated root $m_1 = m_2 = m$	$(c_1 + c_2 x)e^{mx}$	Transient (critically damped)
CF: Complex roots $\alpha \pm i\beta$	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$	Transient (oscillatory)
PI: Undetermined coefficients	$y_p = $ trial form matching $R(x)$	Steady-state response
PI: Variation of parameters	$y_p = u_1 y_1 + u_2 y_2$	For any $R(x)$
General solution	$y = y_c + y_p$	Full behaviour
Particular IVP solution	y with c_1, c_2 determined by ICs	Unique solution

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Overall Summary of Unit 5 (Topics 28–32)

Topic	Core Concept	Key Method/Formula
28. Classification	Homogeneous vs Non-Homogeneous	$y'' + Py' + Qy = 0$ vs $R(x) \neq 0$
29. CF	Solve characteristic equation $am^2 + bm + c = 0$	3 cases: distinct, repeated, complex roots
30. PI: Undetermined Coefficients	Guess form of y_p based on $R(x)$	Modification rule when trial ∈ CF
31. PI: Variation of Parameters	$y_p = u_1 y_1 + u_2 y_2$	u_i found using Wronskian — works for all $R(x)$
32. Complete Solution + IVP	$y = y_c + y_p$, then apply ICs	2 ICs → 2 equations → unique c_1, c_2