VIGNAN'S
FOUNDATION FOR SCIENCE, TECHNOLOGY & RESEARCH
(Deemed to be University) - Estd. u/s 3 of UGC Act 1956

Credits: Avanti Sankalp Program

Unit 4: Differentiation

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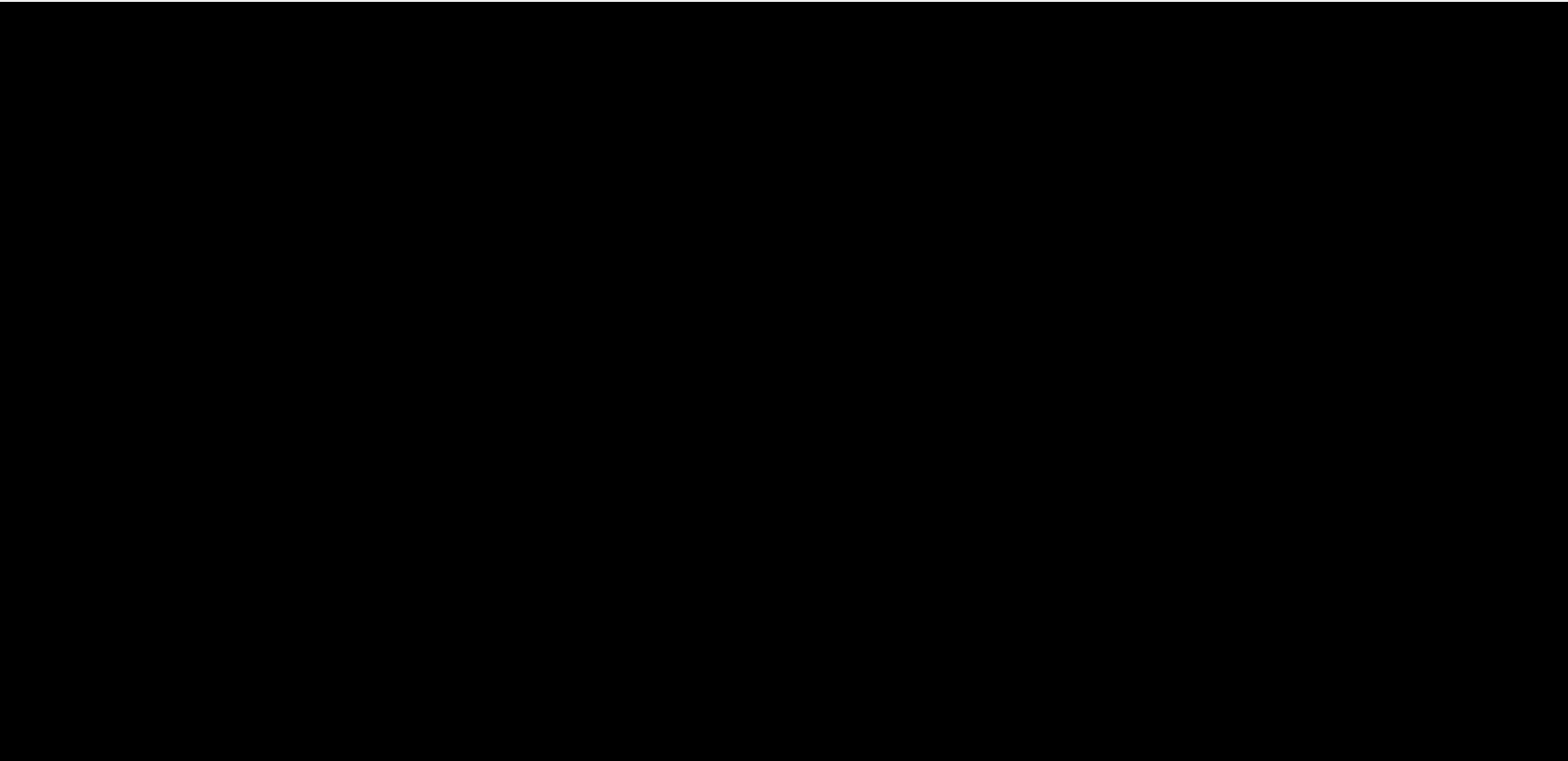
Limits



Limits

Limit	• If $x \to a, f(x) \to L$, then L is called limit of the function $f(x)$. • $\lim_{x \to a} f(x) = L$ • Left Hand Limit of f at $x = a$ is $\lim_{x \to a^-} f(x)$ • Right Hand Limit of f at $x = a$ is $\lim_{x \to a^+} f(x)$ • If the Right and Left Hand Limits coincide The limit of $f(x)$ at $x = a$ and $\lim_{x \to a} f(x)$

Algebra of Limits	• $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$ • $\lim_{x \to a}[f(x) \times g(x)] = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$ • $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ whenever $\lim_{x \to a} g(x) \neq 0$	**Limits of Trigonometric Functions**	• $\lim_{x \to 0} \frac{\sin x}{x} = 1$ • $\lim_{x \to 0} \frac{\tan x}{x} = 1$ • $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example

Question: Find the limits

1. $\lim_{x \to 1} \frac{x-1}{x^2-x}$

2. $\lim_{x \to 0} \frac{\sin 4x}{2x}$

Example

Question: Find the limits

1. $\lim\limits_{x\to 1} \dfrac{x-1}{x^2-x}$

2. $\lim\limits_{x\to 0} \dfrac{\sin 4x}{2x}$

Solution:

1. $\lim\limits_{x\to 1} \dfrac{x-1}{x^2-x}$

$$= \lim_{x\to 1} \frac{x-1}{x(x-1)}$$

$$= \lim_{x\to 1} \frac{1}{x}$$

$$= 1$$

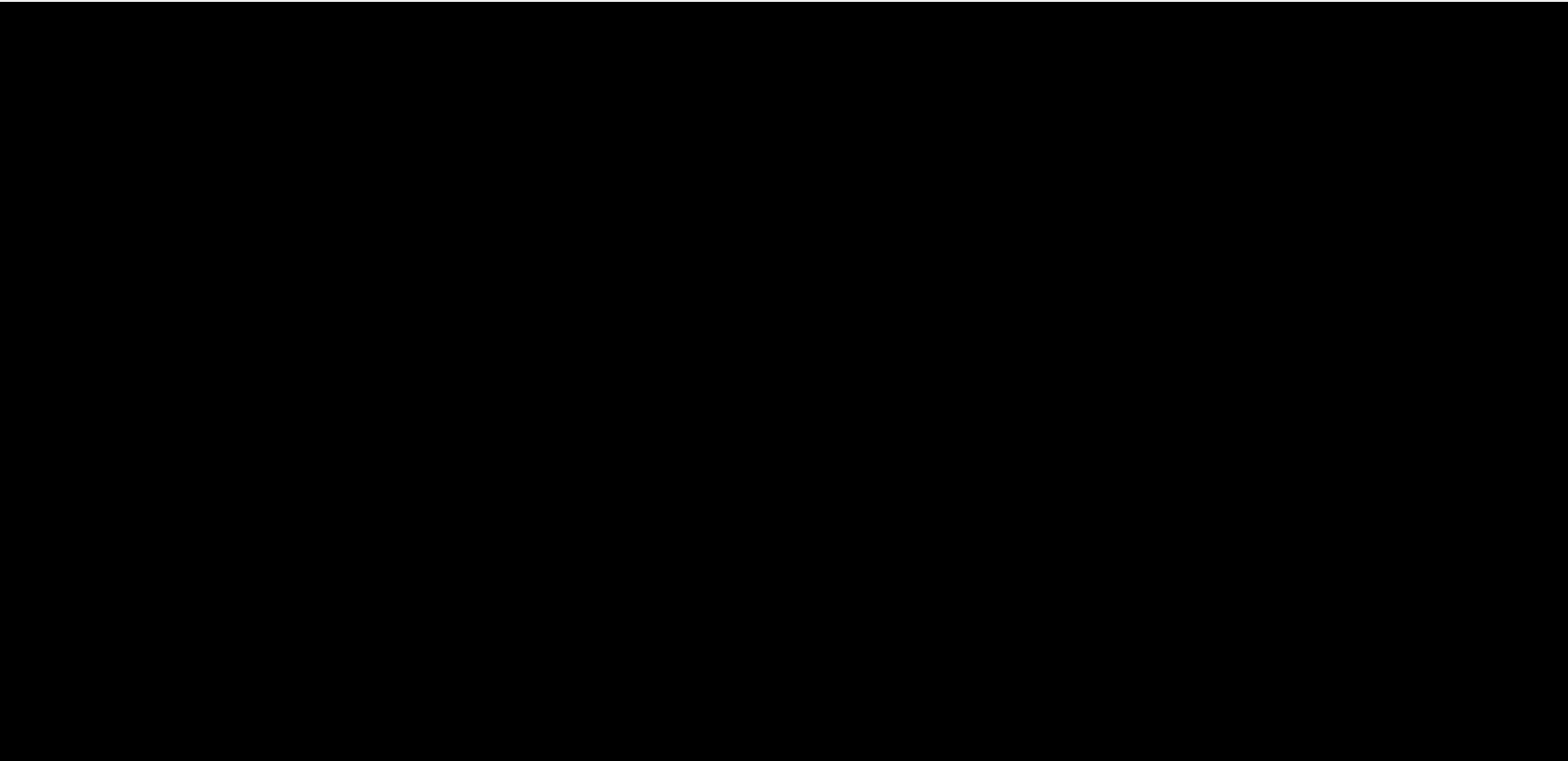
2. $\lim\limits_{x\to 0} \dfrac{\sin 4x}{2x}$

$$= \lim_{x\to 0} \frac{\sin 4x}{4x} \times 2$$

$$= 1 \times 2$$

$$= 2$$

Continuity



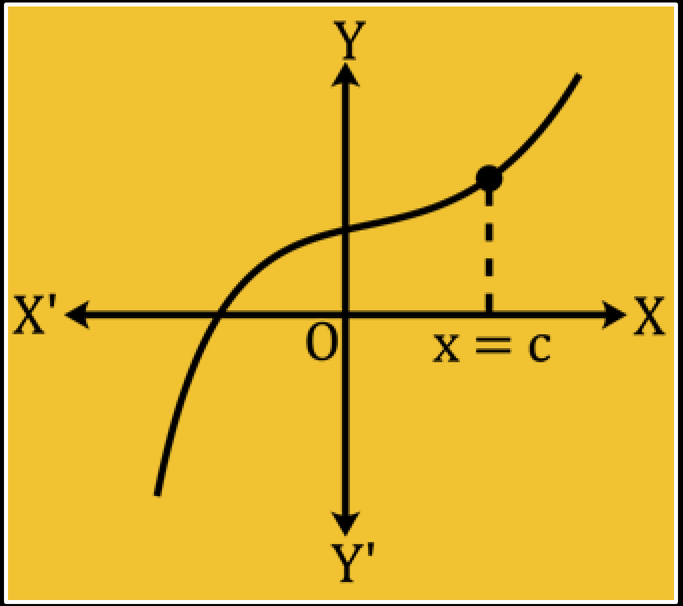
Continuity

Let **f** be a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

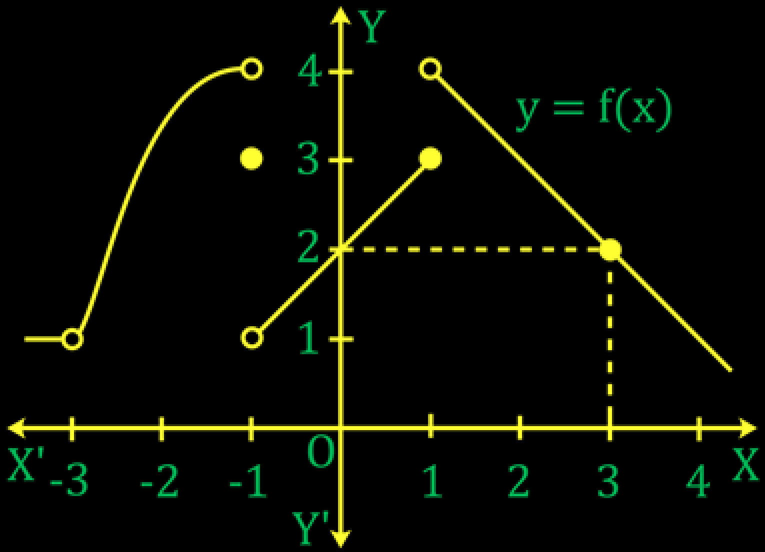
$$\boxed{\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)}$$

If f is not continuous at c ⇒ f is discontinuous at c and c is called a point of discontinuity of f.



Continuity

Example: $y = f(x)$



Continuity

Example: $y = f(x)$

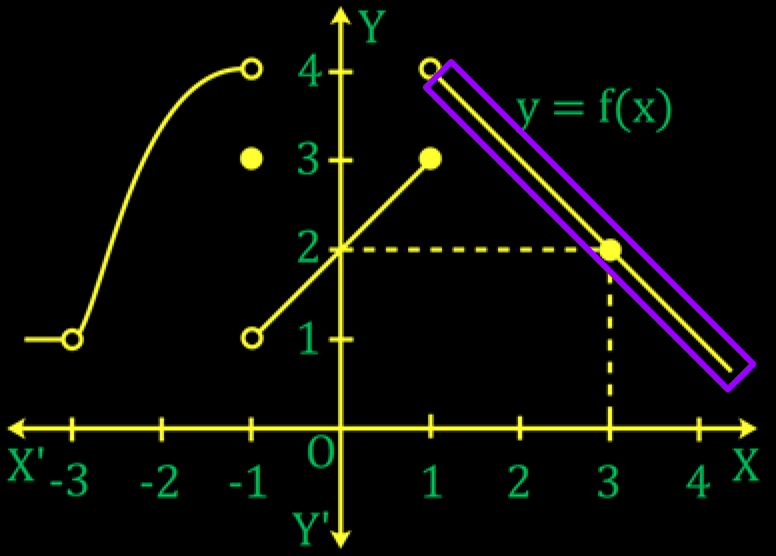
At $x = -3$, $f(-3) = \text{Not Defined}$

$\lim_{x \to -3^-} f(x) = 1 = \lim_{x \to -3^+} f(x)$ $\longrightarrow$ Not Continuous

At $x = -1$, $f(-1) = 3$

$\lim_{x \to -1^-} f(x) = 4$, $\lim_{x \to -1^+} f(x) = 1$ $\longrightarrow$ Not Continuous

$\lim_{x \to -1} f(x) = \text{Not Defined}$



At $x = 1$, $f(1) = 3$

$\lim_{x \to 1^-} f(x) = 3$ and $\lim_{x \to 1^+} f(x) = 4$ $\longrightarrow$ Not Continuous

$f(1) = \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

At $x = 3$, $f(3) = 2$

$\lim_{x \to 3^-} f(x) = 2 = \lim_{x \to 3^+} f(x)$ $\longrightarrow$ Continuous

$\lim_{x \to 3^-} f(x) = f(3) = \lim_{x \to 3^+} f(x)$

Continuity

Question: Check the continuity of the following functions

1. $f(x) = 2x + 3$ at $x = 1$.
2. $f(x) = |x|$ at $x = 0$.

Continuity

Question: Check the continuity of the following functions

1. $f(x) = 2x + 3$ at $x = 1$.
2. $f(x) = |x|$ at $x = 0$.

Solution:

1. Given that $f(x) = 2x + 3$

At $x = 1$

$f(1) = 2(1) + 3 = 5$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1}(2x + 3)$$
$$= 2(1) + 3$$
$$= 5$$

Thus, $\lim_{x \to 1} f(x) = 5 = f(1)$

Hence, f is continuous at $x = 1$.

2. Given $f(x) = |x|$

Now, $f(0) = |0| = 0$

Left hand limit of f at $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h)$$
$$= \lim_{h \to 0} |0 - h| = 0$$

Right hand limit of f at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h)$$
$$= \lim_{h \to 0} |0 + h| = 0$$

Thus, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Hence, f is continuous at $x = 0$.

Continuity - Plots

Question: Check the continuity of the following functions

1. $f(x) = 2x + 3$ at $x = 1$.
2. $f(x) = |x|$ at $x = 0$.

Solution:

1. Given that $f(x) = 2x + 3$
2. Given $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Example

Question: Show that the functions f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Example

Question: Show that the functions f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

Given

is not continuous at $x = 0$.

Solution: The function is defined at $x = 0$

$f(0) = 1$

When $x \neq 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}(x^3 + 3)$$

$$= 0^3 + 3 = 3$$

$$f(0) \neq \lim_{x \to 0} f(x)$$

Hence, f is not continuous at $x = 0$.

and $x = 0$ is the point of discontinuity for this function.

Example

Question: Find the values of a and b so the function $f(x) = \begin{cases} x + a, & \text{if } x < 3 \\ 3, & \text{if } x = 3 \\ 6x + b, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

12M05.1 - Continuity

Question: Find the values of a and b so the function $f(x) = \begin{cases} x + a, & \text{if } x < 3 \\ 3, & \text{if } x = 3 \\ 6x + b, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

Solution: LHL = f(3) = RHL

$$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 - h) = \lim_{h \to 0} (3 - h) + a = \boxed{3 + a}$$

$$f(3) = \boxed{3}$$

$$\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3 + h) = \lim_{h \to 0} 6(3 + h) + b = 6(3) + b = \boxed{18 + b}$$

Equal → $a = 0$

Equal → $b = -15$

Example

Question: Find the values of k so the function $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$.

Example

Question: Find the values of k so the function $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$.

Solution: $f\left(\frac{\pi}{2}\right) = \lim_{x \to \frac{\pi}{2}} f(x)$

$$\Rightarrow 3 = \boxed{\lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x}} \quad \frac{0}{0} \text{ Form}$$

$$\Rightarrow 3 = \frac{k}{2}$$

$$\Rightarrow k = 6$$

Put $x = \frac{\pi}{2} + h$, then

$x \to \frac{\pi}{2} \Rightarrow h \to 0$

$$\therefore \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-k\sin h}{-2h}$$

$$= \frac{k}{2} \times \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \frac{k}{2} \quad \left(\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right)$$

Results

Suppose f and g be two real functions continuous at a real number $x = c$, then

Functions $f + g$, $f - g$, $f \cdot g$, $\boxed{\frac{f}{g}}$ are continuous at $x = c$.

$\boxed{g(c) \neq 0}$

If f and g are two functions, then $(fog)(x) = f(g(x))$.

1. If (fog) is defined at $x = c$.
1. If g is continuous at $x = c$. $\Rightarrow$ $\{ (fog)$ is continuous at $x = c \}$
1. If f is continuous at $x = g(c)$

Example

Question: Is the function defined by $f(x) = x^2 - \sin x$ continuous at $x = \pi$?

12M05.1 - Continuity

Question: Is the function defined by $f(x) = x^2 - \sin x$ continuous at $x = \pi$?

Solution:

Two Methods

Normal Method

$$\lim_{x \to \pi} f(x) = f(\pi)$$

Algebra of continuous Functions

$$f(x) = p(x) \pm q(x)$$

12M05.1 - Continuity

Question: Is the function defined by $f(x) = x^2 - \sin x$ continuous at $x = \pi$?

Solution: Given that $f(x) = x^2 - \sin x$

Let $f(x) = p(x) - q(x)$ where $p(x) = x^2$ and $q(x) = \sin x$

Continuity of $p(x) = x^2$ at $x = \pi$

$p(\pi) = \pi^2$

$$\lim_{x \to \pi} p(x) = \lim_{h \to 0} p(\pi + h)$$
$$= \lim_{h \to 0} (\pi + h)^2$$
$$= (\pi + 0)^2$$
$$= \pi^2$$

$\because p(\pi) = \lim_{x \to \pi} p(x)$

$\therefore p(x) = x^2$ is continuous at $x = \pi$. ...(i)

Continuity of $q(x) = \sin x$ at $x = \pi$

$q(\pi) = \sin \pi = 0$

$$\lim_{x \to \pi} q(x) = \lim_{h \to 0} q(\pi + h)$$
$$= \lim_{h \to 0} \sin(\pi + h)$$
$$= \sin \pi$$
$$= 0$$

$\because q(\pi) = \lim_{x \to \pi} q(x)$

$\therefore q(x) = \sin x$ is continuous at $x = \pi$ ...(ii)

From Eq. (i) and (ii)

$f(x) = p(x) - q(x) = x^2 - \sin x$ is continuous at $x = \pi$.

Question: Show that the function f is defined by $f(x) = |1 - x + |x||$, where x is any real number, is continuous function.

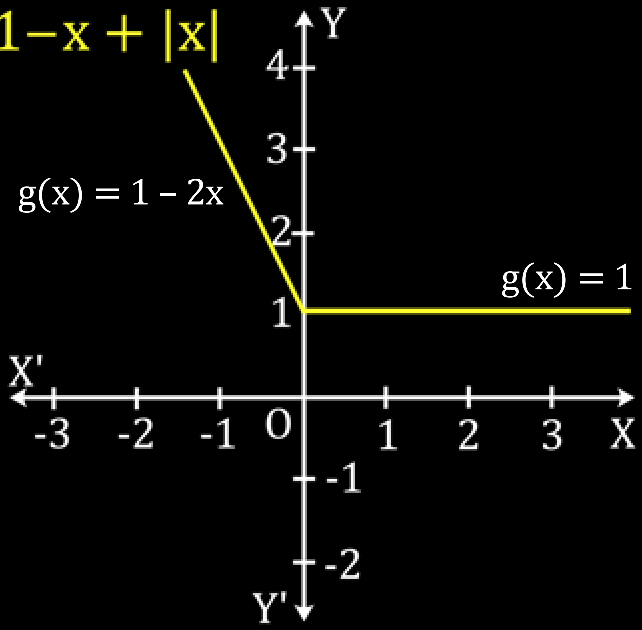
Question: Show that the function f is defined by $f(x) = |1 - x + |x||$, where x is any real number, is continuous function.

Solution: Let $g(x) = 1 - x + |x|$
and $h(x) = |x|$

$f(x) = h(g(x)) = hog(x)$

$$g(x) = \begin{cases} 1 - x + x, & \text{if } x \geq 0 \\ 1 - x - x, & \text{if } x < 0 \end{cases} = \begin{cases} 1, & \text{if } x \geq 0 \\ 1 - 2x, & \text{if } x < 0 \end{cases}$$

$g(x) = 1 - x + |x|$



Summary

1. **Continuity of Function at point x = c**

$$\lim_{x \to c} f(x) = f(c)$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c)$$

1. **Geometrical Meaning of Continuity.**

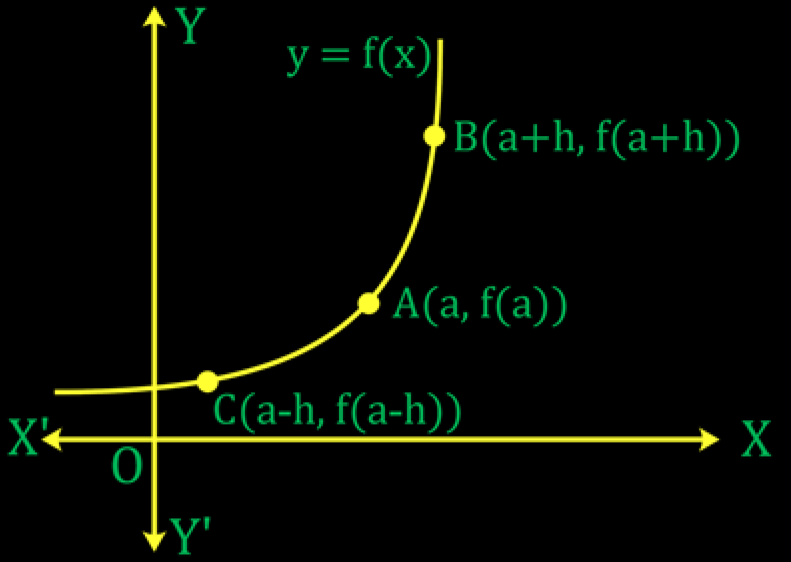
In Graph — No Break or Jump → Continuous Function

In Graph — Any Break or Jump → Discontinuous

Function

Differentiability and Derivatives of Composite Functions

Example: $y = f(x)$



Differentiability and Derivatives of Composite Functions

Example: $y = f(x)$

$$\text{Slope of CA} = \frac{f(a-h)-f(a)}{a-h-a} = \frac{f(a-h)-f(a)}{-h}$$

as $h \to 0 \Rightarrow C \to A$

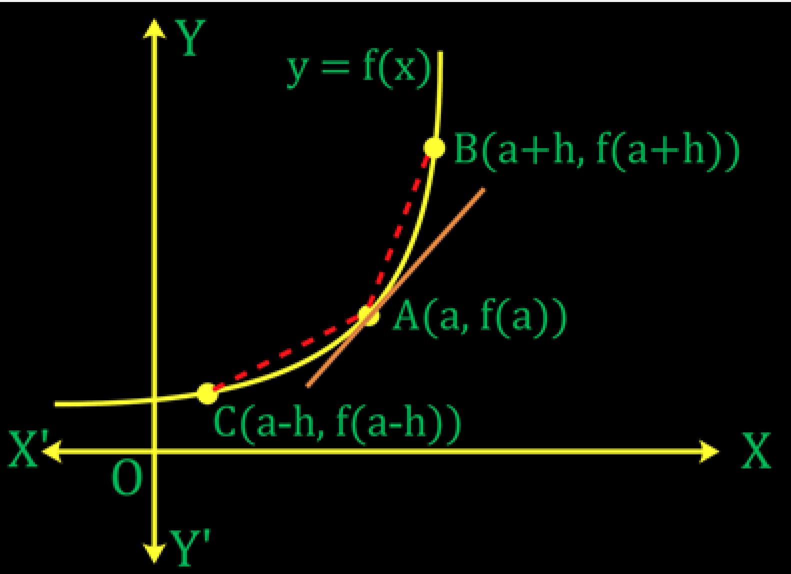
$$\text{Tangent at Point A} = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

$$\text{Slope of AB} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

as $h \to 0 \Rightarrow B \to A$

$$\text{Tangent at Point A} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} \rightarrow \text{Left Hand Derivative (LHD)}$$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \rightarrow \text{Right Hand Derivative (RHD)}$$

Differentiability and Derivatives of Composite Functions

The derivative of a real function	f is a real function and c is a point in its domain, then the derivative of f at $x = c$ is defined by $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ provided this limit exists.
Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))\|_c$	$$f'(x) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$ wherever the limit exists is defined to be the derivative of f.
The process of finding derivative of a function is called **Differentiation**.	
Differentiate $f(x)$ with respect to x to mean find $f'(x)$.	

Differentiability and Derivatives of Composite Functions

Algebra of Derivatives

1. $(u \pm v)' = u' \pm v'$
2. $(uv)' = u'v + uv'$ (Product Rule)
3. $\left(\frac{u}{v}\right)' = \frac{(u'v - uv')}{v^2}$, wherever $v \neq 0$ (Quotient Rule)

Derivatives of Some Standard Functions							
$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\operatorname{cosec} x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$	$-\operatorname{cosec}^2 x$	$\sec x \tan x$	$-\cot x \operatorname{cosec} x$

Example

Question: Find the derivative of

1. $5x^{-2} + 3x^3$
2. $\sin x \cos x$

Example

Question: Find the derivative of

1. $5x^{-2} + 3x^3$
2. $\sin x \cos x$

Solution:

1. Let $y = 5x^{-2} + 3x^3$

$$\frac{dy}{dx} = \frac{d}{dx}(5x^{-2}) + \frac{d}{dx}(3x^3)$$

$$= 5\frac{d}{dx}(x^{-2}) + 3\frac{d}{dx}(x^3)$$

$$= 5(-2x^{-3}) + 3(3x^2)$$

$$= -10x^{-3} + 9x^2$$

2. Let $y = \sin x \cos x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \cos x)$$

By product rule

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x\,(-\sin x) + \cos x\,(\cos x)$$

$$= -\sin^2 x + \cos^2 x$$

$$= -\cos 2x$$

Differentiability and Derivatives of Composite Functions

$$\boxed{\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}} = \boxed{\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

A function f(x) is differentiable in

(a, b)

f(x) is differentiable $\forall\, x \in (a, b)$.

[a, b]

- f(x) is differentiable $\forall\, x \in (a, b)$.
- f(x) is differentiable at the end points a and b, i.e.,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h\to 0}\frac{f(b-h)-f(b)}{-h}$$

exist finitely

Differentiability and Derivatives of Composite Functions

Theorem: If a function f is differentiable at a point c, then it is also continuous at that point.

Differentiability and Derivatives of Composite Functions

Theorem: If a function f is differentiable at a point c, then it is also continuous at that point.

Proof: Since f is differentiable at c, we have

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)$$

But for $x \neq c$, we have

$$f(x)-f(c)=\frac{f(x)-f(c)}{x-c}\times(x-c)$$

$$\Rightarrow \lim_{x\to c}[f(x)-f(c)]=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\times(x-c)\right]$$

$$\Rightarrow \lim_{x\to c}f(x)-\lim_{x\to c}f(c)=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\times\lim_{x\to c}(x-c)$$

$$\Rightarrow \lim_{x\to c}f(x)-f(c)=f'(c)\times 0$$

$$\Rightarrow \lim_{x\to c}f(x)-f(c)=0$$

$$\Rightarrow \lim_{x\to c}f(x)=f(c)$$

Hence f is continuous at $x = c$.

Differentiability and Derivatives of Composite Functions

Every differentiable function is continuous but every continuous function is not differentiable.

Example:

12M05.2 - Differentiability and Derivatives of Composite Functions

Every differentiable function is continuous but every continuous function is not differentiable.

Example:

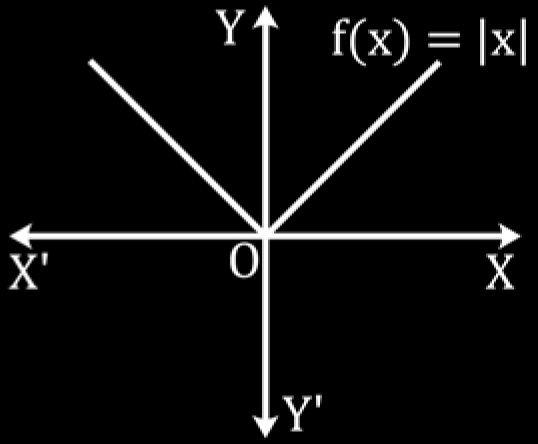
$f(x) = |x|$ is a continuous function.

LHD at $x = 0$

$$\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{|0-h|-0}{-h}$$
$$= \lim_{h\to 0}\frac{|-h|}{-h}$$
$$= \lim_{h\to 0}\frac{h}{-h}$$
$$= \lim_{h\to 0} -1$$
$$= -1$$

RHD at $x = 0$

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|-0}{h}$$
$$= \lim_{h\to 0}\frac{|h|}{h}$$
$$= \lim_{h\to 0}\frac{h}{h}$$
$$= \lim_{h\to 0} 1$$
$$= 1$$



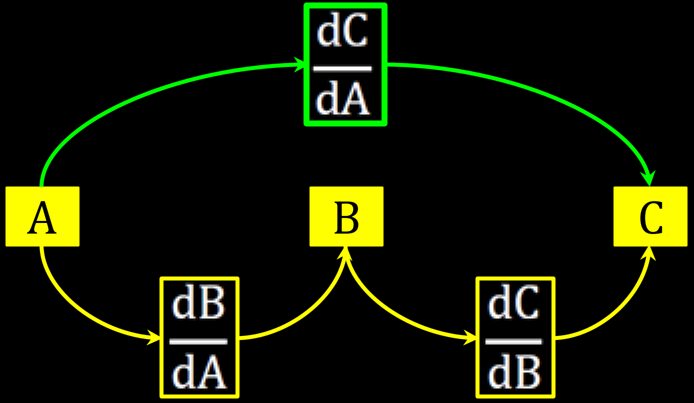
LHD $\neq$ RHD

hence f is not differentiable at 0. Thus, f is not a differentiable function.

Points to Remember

If a function is

Differentiable	→	Definitely Continuous
Continuous	→	Not Necessarily Differentiable
Discontinuous	→	Definitely Not Differentiable



$$\frac{dC}{dA} = \frac{dB}{dA} \times \frac{dC}{dB} \Rightarrow \boxed{\frac{dC}{dA} = \frac{dC}{dB} \times \frac{dB}{dA}}$$

Chain Rule

$$\boxed{\frac{dD}{dA} = \frac{dD}{dC} \times \frac{dC}{dB} \times \frac{dB}{dA}}$$

Differentiability and Derivatives of Composite Functions

Example: $\frac{d}{dx}(\cos 2x)$

Differentiability and Derivatives of Composite Functions

Example: $\boxed{\dfrac{d}{dx}(\cos 2x)} \longrightarrow \boxed{-\sin 2x}$ ✗

$$\boxed{\frac{dC}{dA} = \frac{dC}{dB} \times \frac{dB}{dA}}$$

$A \to x$

$B \to 2x$

$C \to \cos 2x$

$$\frac{d}{dx}(\cos 2x) = \frac{d}{d(2x)}(\cos 2x) \times \frac{d}{dx}(2x)$$

Let $2x =$

$$= \frac{d}{dt}(\cos t) \times 2$$

$$= -\sin t \times 2$$

$$\boxed{= -2\sin 2x}$$

$x \to \cos 2x$ ✗

$x \to 2x \to \cos 2x$

Differentiability and Derivatives of Composite Functions

Question: $\frac{d}{dx}(\cos(\sin x))$

Differentiability and Derivatives of Composite Functions

Question: $\frac{d}{dx}(\cos(\sin x))$

Solution: $x \to \sin x \to \cos(\sin x)$

$A \to x \qquad B \to \sin x \qquad C \to \cos(\sin x)$

$$\frac{dC}{dA} = \frac{dC}{dB} \times \frac{dB}{dA}$$

$$\frac{d}{dx}(\cos(\sin x)) = \boxed{\frac{d}{d\sin x}(\cos(\sin x))} \cdot \frac{d}{dx}(\sin x)$$

Let $\sin x = u$

$$= \boxed{\frac{d}{du}(\cos u)} \cdot \cos x$$

$$= -\sin u \cdot \cos x$$

Substituting the value of u

$$= -\sin(\cos x) \cdot \cos x$$

Differentiability and Derivatives of Composite Functions

Question: $\frac{d}{dx}\left(\sec\left(\tan\left(\sqrt{x}\right)\right)\right)$

Differentiability and Derivatives of Composite Functions

Question: $\frac{d}{dx}\left(\sec\left(\tan\left(\boxed{\sqrt{x}}\right)\right)\right)$

$$\boxed{\frac{dD}{dA} = \frac{dD}{dC} \times \frac{dC}{dB} \times \frac{dB}{dA}}$$

Solution: $x \to \sqrt{x} \to \tan\left(\sqrt{x}\right) \to \sec\left(\tan\left(\sqrt{x}\right)\right)$

$A \to x \qquad B \to \sqrt{x} \qquad C \to \tan\sqrt{x} \qquad D \to \sec\left(\tan\sqrt{x}\right)$

$$\frac{d}{dx}\left(\sec\left(\tan\left(\sqrt{x}\right)\right)\right) = \frac{d}{d\tan\left(\sqrt{x}\right)}\left(\sec\left(\tan\left(\sqrt{x}\right)\right)\right) \times \frac{d}{d\sqrt{x}}\left(\tan\sqrt{x}\right) \times \frac{d}{dx}\left(\sqrt{x}\right)$$

Let $\tan\left(\sqrt{x}\right) = u$ and $\sqrt{x} = v$

$$= \frac{d}{du}(\sec u) \times \frac{d}{dv}(\tan v) \times \frac{d}{dx}\left(x^{1/2}\right)$$

$$= \sec u \tan u \times \sec^2 v \times \frac{1}{2\sqrt{x}}$$

$$= \sec\left(\tan\left(\sqrt{x}\right)\right)\tan\left(\tan\left(\sqrt{x}\right)\right) \times \sec^2\left(\sqrt{x}\right) \times \boxed{\frac{1}{2\sqrt{x}}}$$

Differentiability and Derivatives of Composite Functions

Question: $\frac{d}{dx}\left(2\sqrt{\cot(x^2)}\right)$

Differentiability and Derivatives of Composite Functions

Question: $\frac{d}{dx}\left(2\sqrt{\cot(x^2)}\right)$

$$\frac{dD}{dA} = \frac{dD}{dC} \times \frac{dC}{dB} \times \frac{dB}{dA}$$

Solution: $\frac{d}{dx}\left(2\sqrt{\cot(x^2)}\right) = 2 \times \frac{d}{dx}\left(\sqrt{\cot(x^2)}\right)$

$$x \to x^2 \to \cot(x^2) \to \sqrt{\cot(x^2)}$$

$A \to x \qquad B \to x^2 \qquad C \to \cot(x^2) \qquad D \to \sqrt{\cot(x^2)}$

$$\frac{d}{dx}\left(\sqrt{\cot(x^2)}\right) = \frac{d}{d\cot(x^2)}\left(\sqrt{\cot(x^2)}\right) \cdot \frac{d}{dx^2}(\cot(x^2)) \cdot \frac{d}{dx}(x^2)$$

Let $\cot(x^2) = u$ and $x^2 = v$

$$= \frac{d}{du}(\sqrt{u}) \cdot \frac{d}{dv}(\cot v) \cdot 2x$$

$$= \frac{1}{2\sqrt{u}} \cdot (-\operatorname{cosec}^2 v) \cdot 2x \qquad = -\frac{1}{\sqrt{\cot(x^2)}} \cdot \operatorname{cosec}^2(x^2) \cdot x$$

$$\frac{d}{dx}\left(2\sqrt{\cot(x^2)}\right) = 2 \times -\frac{x \cdot \operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}} = -\frac{2x \cdot \operatorname{cosec}^2(x^2)}{\sqrt{\cot(x^2)}}$$

Summary

Differentiability of Standard Functions

All of the standard functions are differentiable except at certain points, as follows:

1. **Polynomial functions** are **differentiable** in its **domain(R).**
1. A **rational function** $\frac{p(x)}{q(x)}$ is **differentiable except where $q(x) = 0$**, where the function grows to infinity.

   E.g. $\frac{1}{x}$ and $\frac{1}{x^2}$ both functions are **not differentiable at $x = 0$**.
1. **Sines, cosines** and **exponents** are **differentiable everywhere**

   Tangents and **secants** are **not differentiable** at values where they are not defined, i.e.,

   $$\boxed{x = (2n+1)\frac{\pi}{2}, n \in Z}$$

   Cotangents and **cosecants** are **not differentiable** at values where they are not defined, i.e.,

   $$\boxed{x = n\pi, n \in Z}$$

Summary

1. **Differentiability of Function $f(x)$ at point $x = a$**

$$\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$\Rightarrow$ Left Hand Derivative(LHD) = Right Hand Derivative(RHD)

Geometrical Meaning:

If the curve of function f(x) has **no break point/no sharp edge**, then the function is differentiable.

2. **Derivative of Composite Functions**

Break the given function into form of A, B, C, D, ...

$$\frac{dC}{dA} = \frac{dC}{dB}\times\frac{dB}{dA} \quad \text{or} \quad \frac{dD}{dA} = \frac{dD}{dC}\times\frac{dC}{dB}\times\frac{dB}{dA} \quad \text{or} \quad \ldots$$

Derivatives of Inverse Trigonometric Functions

$f(x)$	$f'(x)$	Domain of f'
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	R
$\cot^{-1} x$	$-\frac{1}{1+x^2}$	R
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, \infty)$
$\text{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, \infty)$

Summary

Important Substitutions for Inverse Trigonometric Function			
Form	Substitution	Form	Substitution
$2x\sqrt{1-x^2}$	$x = \sin\theta$ or $x = \cos\theta$	$4x^3 - 3x$	$x = \cos\theta$
$1 - 2x^2$	$x = \sin\theta$	$\frac{2x}{1+x^2}$	$x = \tan\theta$
$2x^2 - 1$	$x = \cos\theta$	$\frac{1-x^2}{1+x^2}$	$x = \tan\theta$
$3x - 4x^3$	$x = \sin\theta$	$\frac{2x}{1-x^2}$	$x = \tan\theta$

Summary

If the given expression is implicit function

1. Directly differentiate with respect to x
2. Separate like and unlike terms
3. Solve for **dy/dx**

Derivatives of Inverse Trigonometric Function

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sec^{-1} x$	$\dfrac{1}{x\sqrt{1-x^2}}$	$\operatorname{cosec}^{-1} x$	$-\dfrac{1}{x\sqrt{1-x^2}}$

Second Order Derivative and Mean Value Theorem

$$
\begin{tikzcd}
y = f(x) \arrow[d, "\text{Diff. w.r.t } x"] & \\
\dfrac{dy}{dx} \text{ or } y_1 \text{ or } y' \text{ or } f'(x) \arrow[r] \arrow[d, "\text{Again Diff. w.r.t } x"] & \text{1st Order Derivative} \\
\dfrac{d^2y}{dx^2} \text{ or } y_2 \text{ or } y'' \text{ or } f''(x) \arrow[r] & \text{2nd Order Derivative}
\end{tikzcd}
$$

Second Order Derivative and Mean Value Theorem

Example: Find $\frac{d^2y}{dx^2}$ if $y = e^x \sin 5x$.

Second Order Derivative and Mean Value Theorem

Example: Find $\frac{d^2y}{dx^2}$ if $\boxed{y = e^x \sin 5x.}$

Diff. w.r.t x

Solution: $\frac{d}{dx}(y) = \frac{d}{dx}(e^x \sin 5x) = e^x \sin 5x + 5e^x \cos 5x$

$$\Rightarrow \frac{dy}{dx} = e^x(\sin 5x + 5 \cos 5x)$$

Again diff. w.r.t x

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^x(\sin 5x + 5 \cos 5x)\right)$$

Product Rule

$$\Rightarrow \frac{d^2y}{dx^2} = (\sin 5x + 5 \cos 5x)\frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\sin 5x + 5 \cos 5x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x(\sin 5x + 5 \cos 5x) + e^x(5 \cos 5x - 25 \sin 5x)$$

Simplification

$$\Rightarrow \frac{d^2y}{dx^2} = 2e^x(5 \cos 5x - 12 \sin 5x)$$

Second Order Derivative and Mean Value Theorem

Question: If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in term of y alone.

Second Order Derivative and Mean Value Theorem

Question: If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in term of y alone.

Solution: $\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x)$

Diff. w.r.t x

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Diff. w.r.t x

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{d}{dx}\left((1-x^2)^{-1/2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left(-\frac{1}{2}\cdot(1-x^2)^{-3/2}\cdot -2x\right)$$

Chain Rule

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{x}{(1-x^2)^{3/2}}$$

$\because y = \cos^{-1} x \Rightarrow x = \cos y$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\cos y}{(1-\cos^2 y)^{3/2}} = -\frac{\cos y}{(\sin y)^{3/2}} = -\frac{\cos y}{\sin^3 y} = -\cot y \operatorname{cosec}^2 y$$

Second Order Derivative and Mean Value Theorem

Question: If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in term of y alone.

Solution:

Second Order Derivative and Mean Value Theorem

Question: If $\boxed{y = \cos^{-1} x}$, find $\boxed{\frac{d^2y}{dx^2} \text{ in term of } y \text{ alone.}}$

Solution: $\cos y = x$

Diff. w.r.t x

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

Chain Rule

$$\Rightarrow -\sin y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} y$$

Diff. w.r.t x

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y) = -\left(-\cot y \operatorname{cosec} y \frac{dy}{dx}\right)$$

Chain Rule

Simplification

$$\Rightarrow \frac{d^2y}{dx^2} = \cot y \operatorname{cosec} y \cdot (-\operatorname{cosec} y) = -\cot y \operatorname{cosec}^2 y$$

Question: If $e^y(x+1)=1$, show that $\dfrac{d^2y}{dx^2}=\left(\dfrac{dy}{dx}\right)^2$.

Question: If $\boxed{e^y(x+1)=1}$, show that $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$.

Solution: $e^y=\frac{1}{x+1}$

Taking Logarithm

$$\Rightarrow y=\log\left(\frac{1}{x+1}\right)$$

$$=\log 1-\log(x+1)$$

$$\Rightarrow y=-\log(x+1)$$

Diff. w.r.t x

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{x+1}$$

Diff. w.r.t x

$$\Rightarrow \frac{d^2y}{dx^2}=-\left(-\frac{1}{(x+1)^2}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(-\frac{1}{x+1}\right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

Second Order Derivative and Mean Value Theorem

Mean Value Theorem

Rolle's Theorem

Lagrange's MVT

$f : [a, b] \to R$

Second Order Derivative and Mean Value Theorem

Mean Value Theorem

Rolle's Theorem

Lagrange's MVT

$f : [a, b] \to R$

f is continuous on [a, b]

f is differentiable on (a, b)

Second Order Derivative and Mean Value Theorem

Mean Value Theorem

Rolle's Theorem

Lagrange's MVT

$f : [a, b] \to R$

If $f(a) = f(b)$, then
$c \in (a, b)$
$f'(c) = 0$

f is continuous on $[a, b]$

f is differentiable on (a, b)

Second Order Derivative and Mean Value Theorem

Mean Value Theorem

Rolle's Theorem

Lagrange's MVT

$f : [a, b] \to R$

If $f(a) = f(b)$, then
$c \in (a, b)$
$f'(c) = 0$

f is **continuous** on $[a, b]$

f is **differentiable** on (a, b)

$c \in (a, b)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Second Order Derivative and Mean Value Theorem

Mean Value Theorem

- **Rolle's Theorem**
- **Lagrange's MVT**

$f : [a, b] \to R$

f is continuous on $[a, b]$

f is differentiable on (a, b)

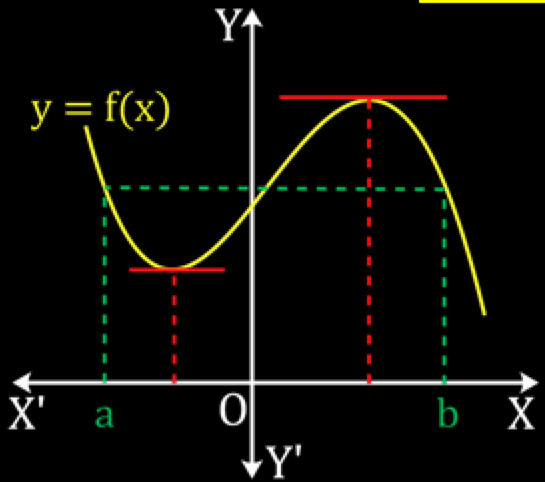
Rolle's Theorem:

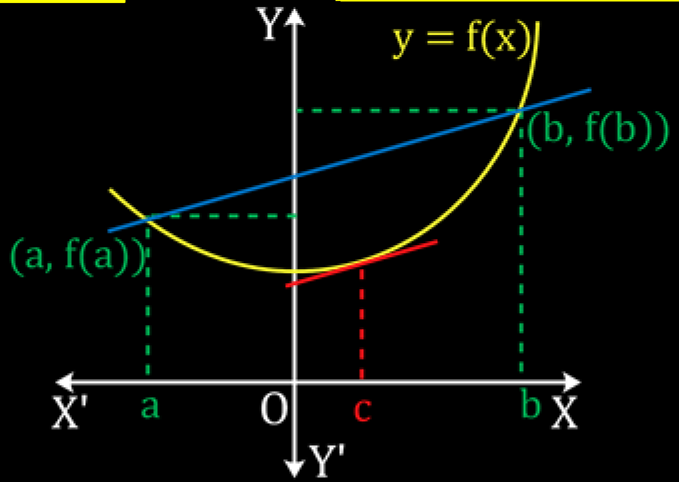
If $f(a) = f(b)$, then
$c \in (a, b)$
$f'(c) = 0$

Lagrange's MVT:

$c \in (a, b)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$





Second Order Derivative and Mean Value Theorem

Example: Verify Rolle's theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Second Order Derivative and Mean Value Theorem

Example: Verify Rolle's theorem for the function $\boxed{f(x) = x^2 + 2x - 8}$, $x \in [-4, 2]$.

Polynomial Function

Continuous on $[-4, 2]$

Differentiable on $(-4, 2)$

Solution: $f(-4) = (-4)^2 + 2(-4) - 8 = 0$

$f(2) = (2)^2 + 2(2) - 8 = 0$

$\because f(-4) = f(2) = 0$

$\therefore$ From Rolle's Theorem

$c \in (-4, 2)$ such that $\boxed{f'(c) = 0}$

$\because f(x) = x^2 + 2x - 8$

$\therefore f'(x) = 2x + 2$

$\boxed{\therefore f'(c) = 2c + 2 = 0}$

$\Rightarrow c = -1 \in (-4, 2)$

Second Order Derivative and Mean Value Theorem

Question: If $f : [-5, 5] \to R$ is a differentiable function and $f'(x)$ does not vanish anywhere, then prove that $f(5) \neq f(-5)$.

Second Order Derivative and Mean Value Theorem

Question: If $f : [-5, 5] \to R$ is a differentiable function and $f'(x)$ does not vanish anywhere, then prove that $f(5) \neq f(-5)$.

$f'(c) \neq 0$

Continuous on $[-5, 5]$

Differentiable on $(-5, 5)$

Solution: By Mean Value Theorem

$c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10f'(c) = f(5) - f(-5)$$

$$\therefore 10f'(c) \neq 0$$

$$\Rightarrow f(5) - f(-5) \neq 0$$

$$\Rightarrow f(5) \neq f(-5)$$

Second Order Derivative and Mean Value Theorem

Question: Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Second Order Derivative and Mean Value Theorem

Question: Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Polynomial Function
Continuous on [1, 3]
Differentiable on (1, 3)

Solution: $f(1) = (1)^3 - 5(1)^2 - 3(1) = -7$

$f(3) = (3)^3 - 5(3)^2 - 3(3) = 27$

$$\frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(-5)}{3 - 1} = \frac{27 - (-7)}{2} = -10$$

From Mean Value Theorem $c \in (1, 3)$ such that

$f(x) = x^3 - 5x^2 - 3x$
$\Rightarrow f'(x) = 3x^2 - 10x - 3$

$f'(c) = -10$

$\Rightarrow 3c^2 - 10c - 3 = -10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow (c - 1)(3c - 7) = 0$ **Factorisation**

$\Rightarrow \boxed{c = 1, 7/3}$

$c = 7/3 \in (1, 3)$ is the only point for which $f'(c) = 0$.

Summary

$$y = f(x) \xrightarrow{\text{Diff. w.r.t } x} \frac{dy}{dx} \text{ or } y_1 \text{ or } y' \text{ or } f'(x) \xrightarrow{\text{Diff. w.r.t } x} \frac{d^2y}{dx^2} \text{ or } y_2 \text{ or } y'' \text{ or } f''(x)$$

$\frac{dy}{dx}$ or y_1 or y' or $f'(x)$ → 1st Order Derivative

$\frac{d^2y}{dx^2}$ or y_2 or y'' or $f''(x)$ → 2nd Order Derivative

Mean Value Theorem

Rolle's Theorem | **Lagrange's MVT**

$f : [a, b] \to R$

f is **continuous** on $[a, b]$

f is **differentiable** on (a, b)

Rolle's Theorem:

If $f(a) = f(b)$, then
$c \in (a, b)$
$f'(c) = 0$

Lagrange's MVT:

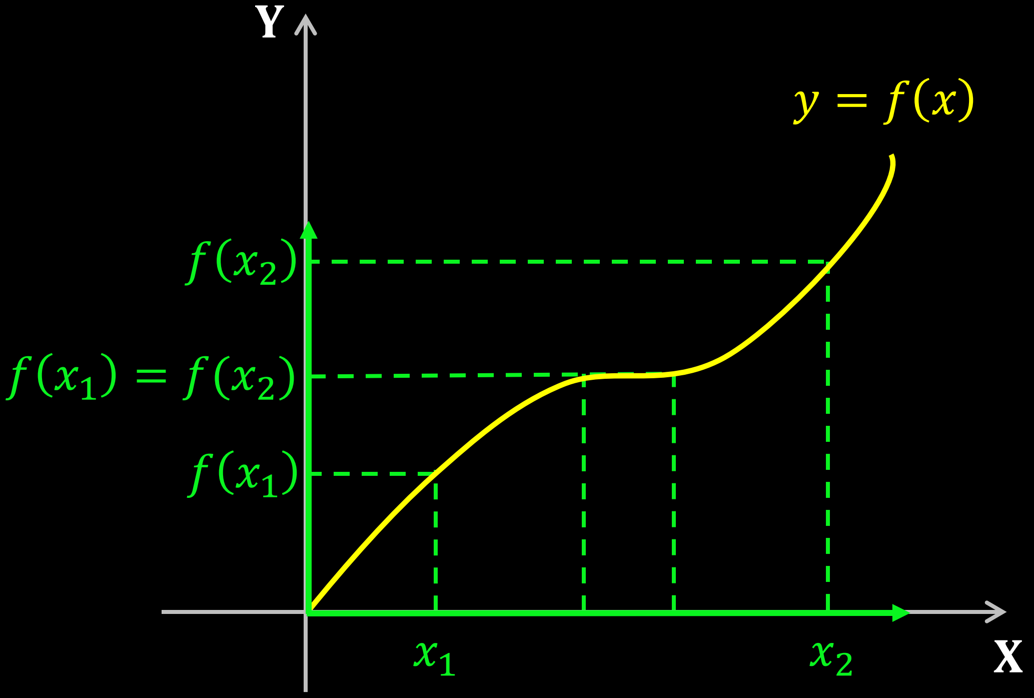
$c \in (a, b)$

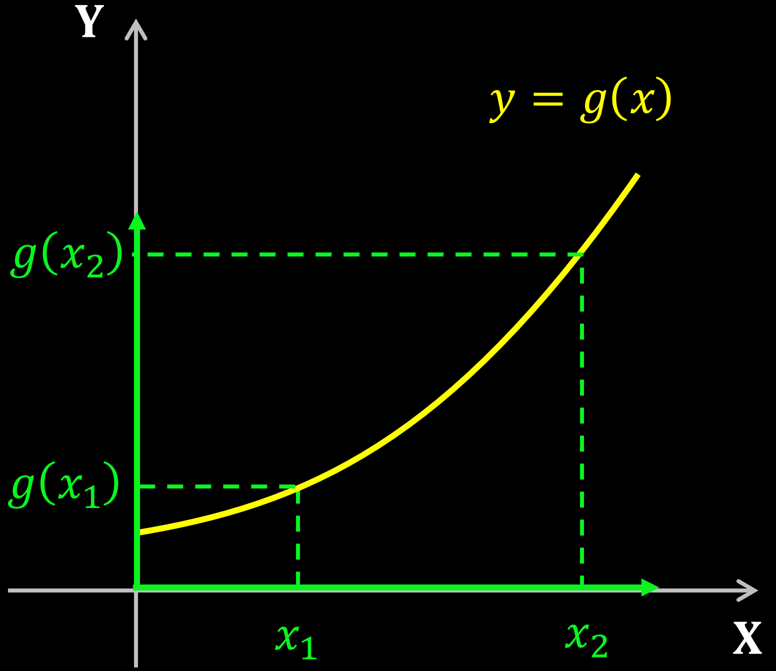
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Chain Rule

$$\begin{tikzcd}[column sep=huge] \boxed{A} \arrow[r, "\frac{dB}{dA}"] \arrow[rr, bend right, "\frac{dC}{dA}"'] & \boxed{B} \arrow[r, "\frac{dC}{dB}"] & \boxed{C} \end{tikzcd}$$

$$\boxed{\frac{dC}{dA} = \frac{dC}{dB} \times \frac{dB}{dA}}$$





Increasing

If $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

Strictly Increasing

If $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$

Ex. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on R.

Sol. Let $x_1, x_2 \in R$

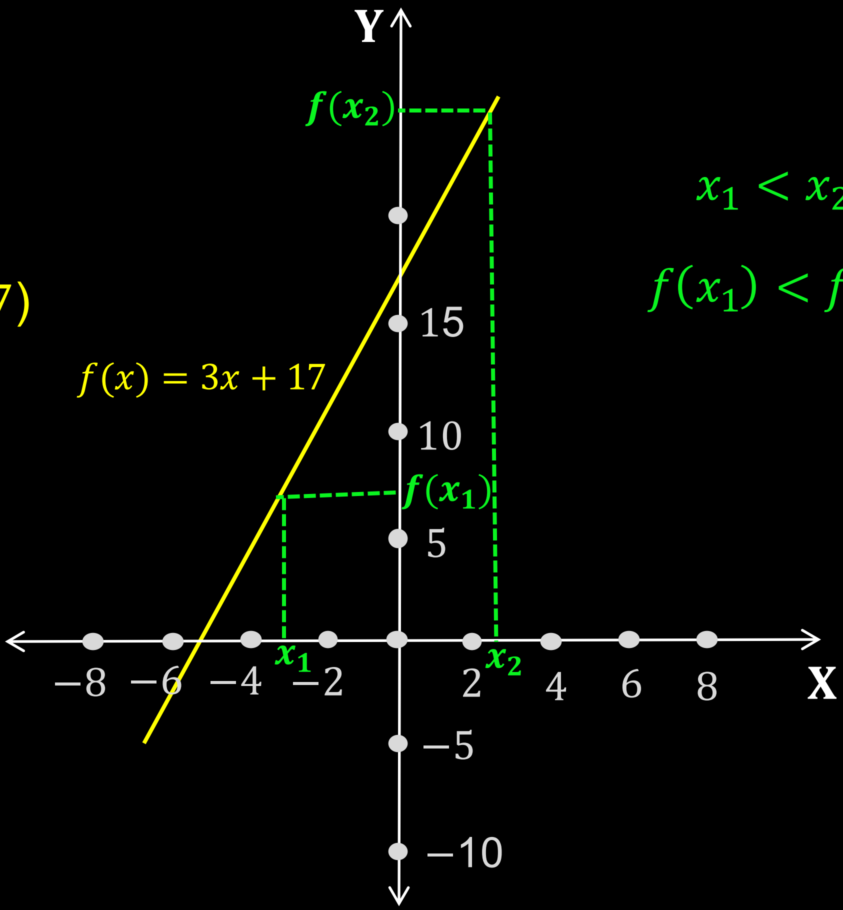
and $x_1 < x_2$

$\Rightarrow 3x_1 < 3x_2$ Multiply by 3

$\Rightarrow 3x_1 + 17 < 3x_2 + 17$ (Adding 17)

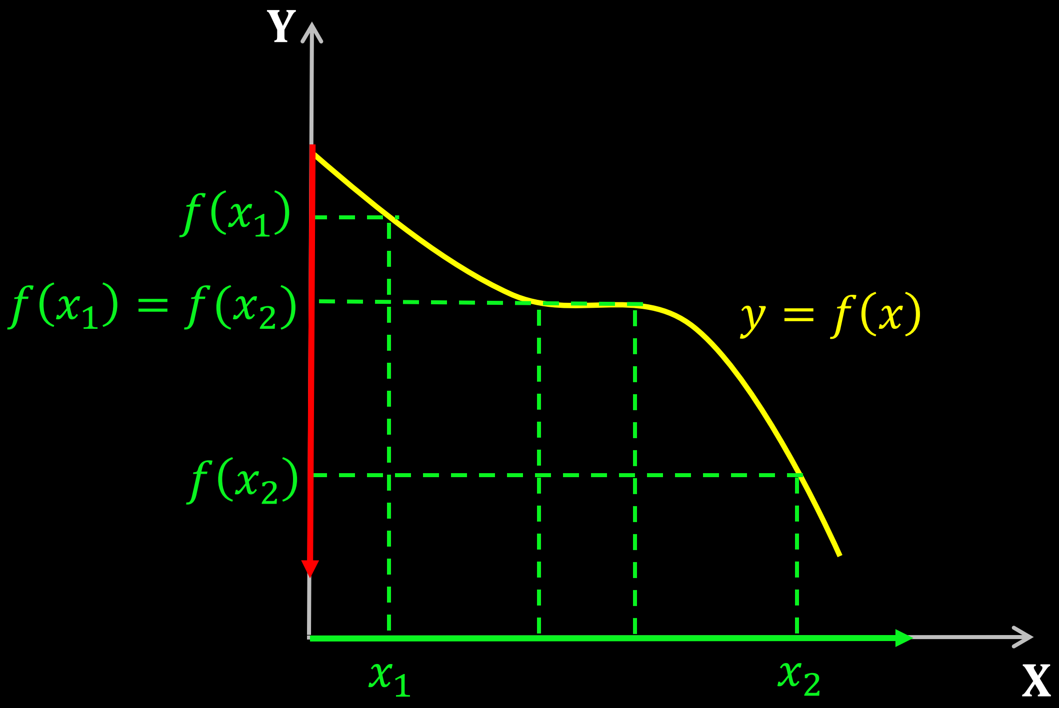
$\Rightarrow f(x_1) < f(x_2)$

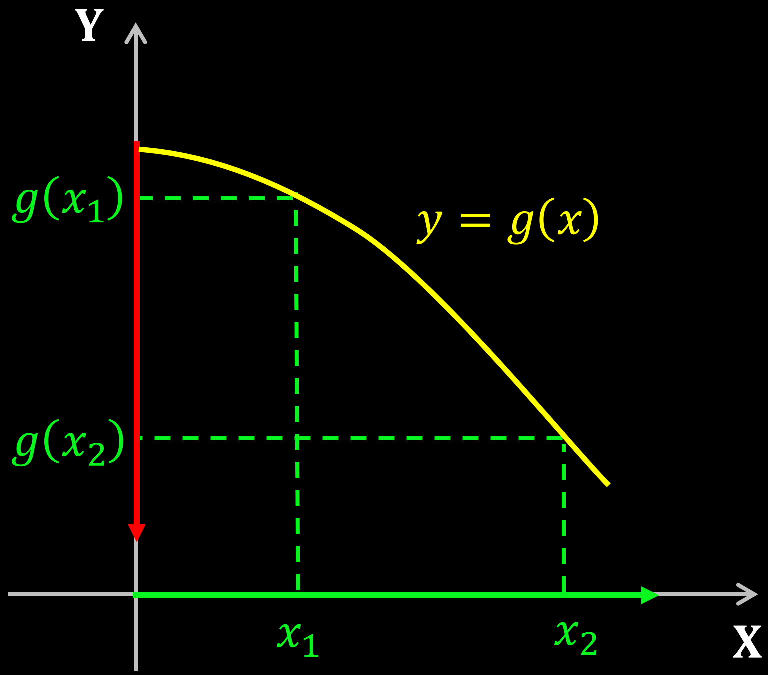
$\therefore f(x)$ is strictly increasing on R



$x_1 < x_2$

$f(x_1) < f(x_2)$





Decreasing

If $x_1 < x_2 \Rightarrow \; f(x_1) \geq f(x_2)$

Strictly Decreasing

If $x_1 < x_2 \Rightarrow \; g(x_1) > g(x_2)$

Ex. Show that the function given by $f(x) = 7 - x$ is strictly decreasing on R.

Sol. Let $x_1, x_2 \in R$

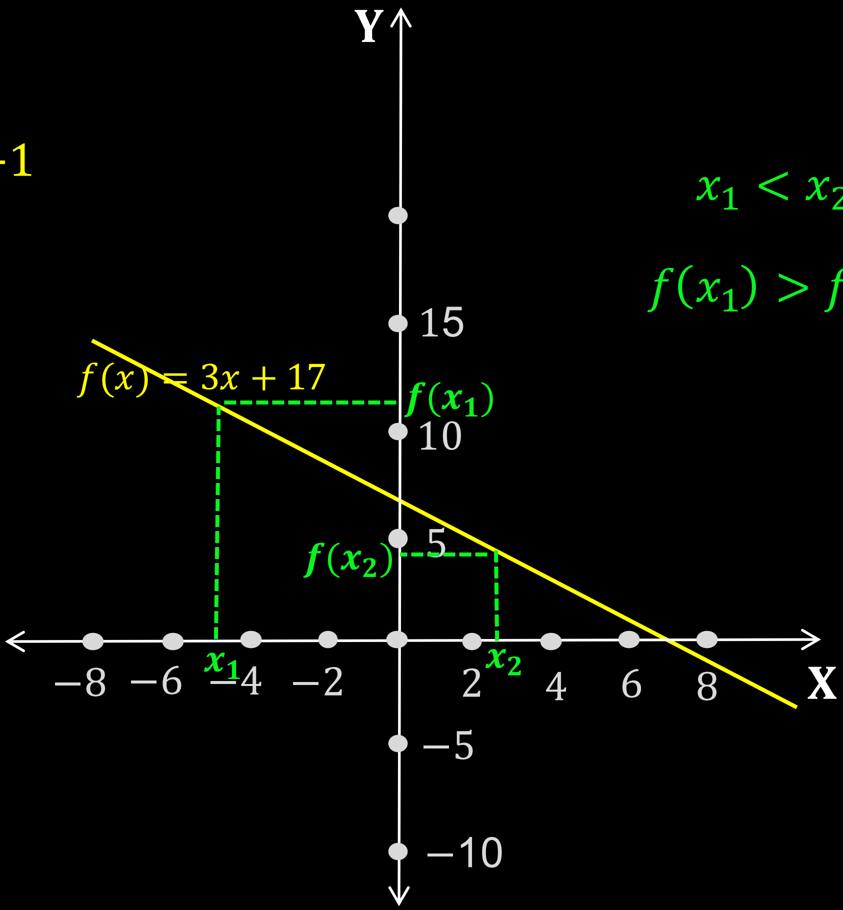
and $x_1 < x_2$

$\Rightarrow -x_1 > -x_2$ Multiply by -1

$\Rightarrow -x_1 + 7 > -x_2 + 7$ (Adding 7)

$\Rightarrow f(x_1) > f(x_2)$

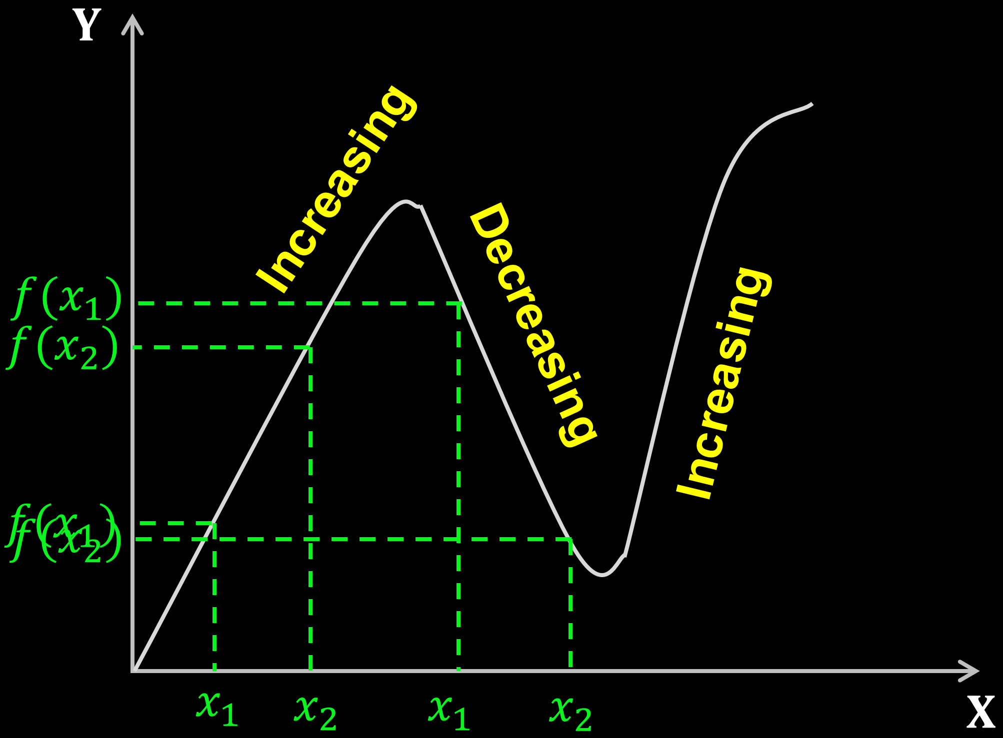
$\therefore f(x)$ is strictly decreasing on R



$x_1 < x_2$

$f(x_1) > f(x_2)$

Neither Increasing nor Decreasing Functions



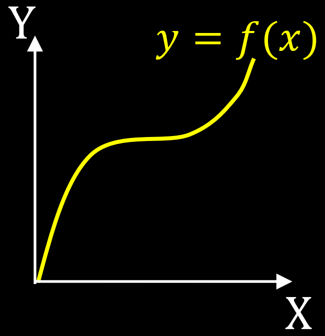
$x_1 < x_2$

$f(x_1) < f(x_2)$

$f(x_1) > f(x_2)$

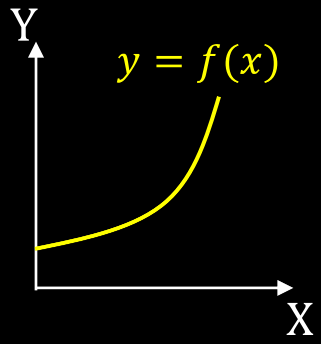
Neither Increasing nor Decreasing Function

Increasing Functions



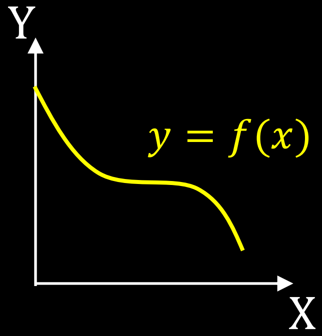
$$f(x_1) \leq f(x_2)$$

Strictly Increasing Functions



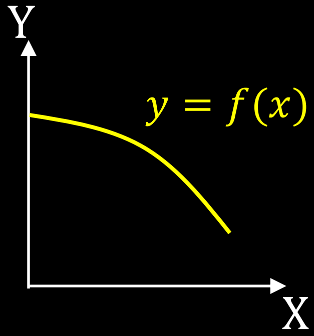
$$f(x_1) < f(x_2)$$

Decreasing Functions



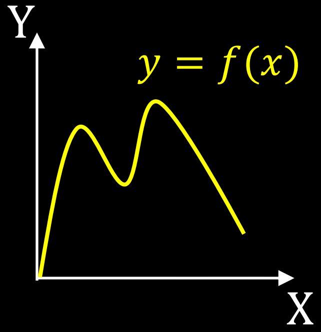
$$f(x_1) \geq f(x_2)$$

Strictly Decreasing Functions



$$f(x_1) > f(x_2)$$

Neither Increasing nor Decreasing Functions



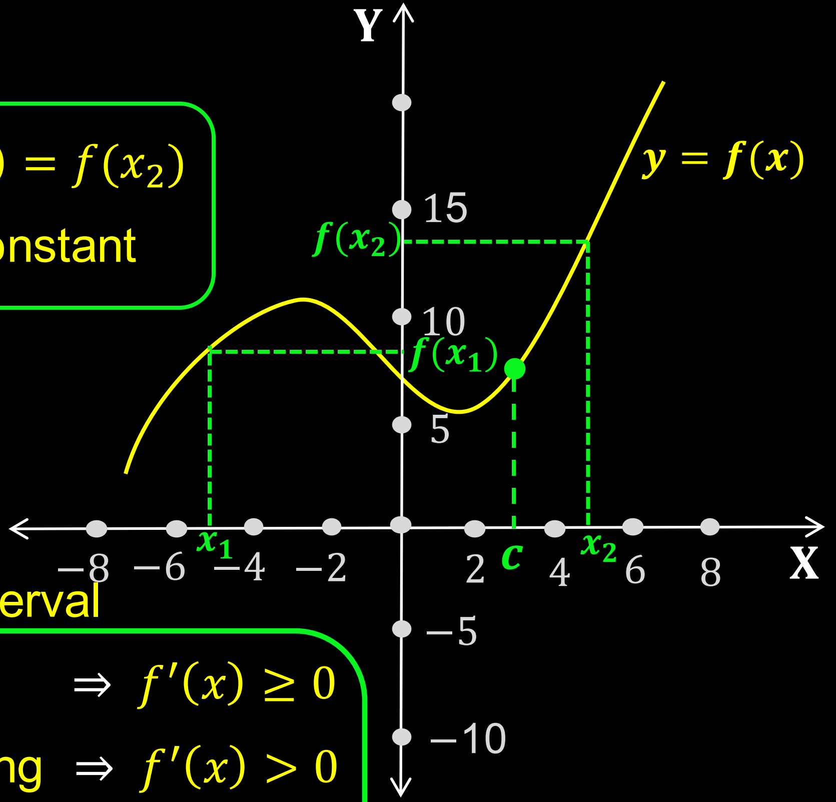
$x_1 < x_2$

$f(x_1) < f(x_2) \Rightarrow$ Increasing

$f(x_1) > f(x_2) \Rightarrow$ Decreasing

$f(x_1) = f(x_2)$

Constant



Mean Value Theorem

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

At a Point

Increasing $\Rightarrow f'(c) > 0$

Decreasing $\Rightarrow f'(c) < 0$

Constant $\Rightarrow f'(c) = 0$

In an Interval

Increasing $\Rightarrow f'(x) \geq 0$

Strictly Increasing $\Rightarrow f'(x) > 0$

Decreasing $\Rightarrow f'(x) \leq 0$

Strictly Decreasing $\Rightarrow f'(x) < 0$

Constant $\Rightarrow f'(x) = 0$

Example: Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Solution: $f(x)$ Increasing $\Rightarrow$ $f'(x) \geq 0$

$$f'(x) = 3x^2 - 6x + 3$$

$$= 3(x^2 - 2x + 1)$$

$$= 3(x - 1)^2$$

Question: Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function on x throughout its domain.

Solution: $\boxed{\text{Increasing} \Rightarrow \frac{dy}{dx} \geq 0}$

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x} - \frac{\boxed{(2+x)\times 2 - 2x}}{(2+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

Taking LCM

$$\Rightarrow \frac{dy}{dx} = \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{(1+x)(2+x)^2} \geq 0$$

Question: Prove that the function f given by $f(x) = \log \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution: $f'(x) = \frac{\cos x}{\sin x} = \cot x$

Question: Find the intervals in which function f is given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is –

A. Increasing

B. Decreasing

Solution: $f'(x) = 6x^2 - 6x - 36$

Question: for what value of a function f is given by $f(x) = x^2 + ax + 1$ is increasing on $[1, 2]$.

Solution:

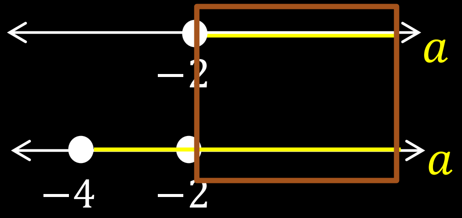
$f'(x) = 2x + a$

$f'(x) > 0$

$\Rightarrow 2x + a \geq 0$

$\Rightarrow a \geq -2x$

$$\because x \in [1,2] \begin{cases} \text{When } x = 1 \quad ; \quad a \geq -2 \\ \text{When } x = 2 \quad ; \quad a \geq -4 \end{cases}$$



$$\boxed{\because a \in [-2, \infty)}$$

Did You Know ?

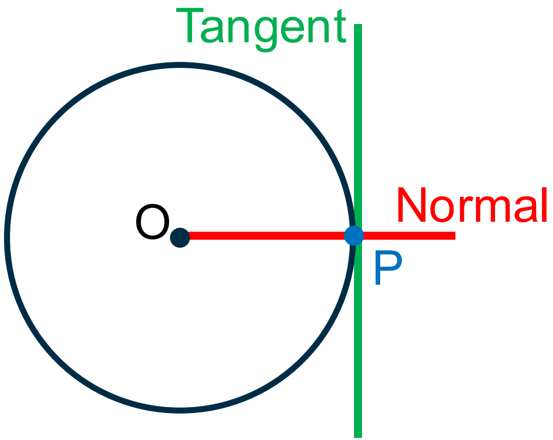
Equation of a Straight Line

$$y - y_0 = m(x - x_0)$$

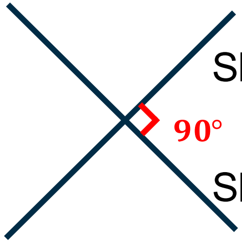
Fixed Point

Slope

Special Cases for Two Lines



Perpendicular



Slope $= m_1$

Slope $= m_2$

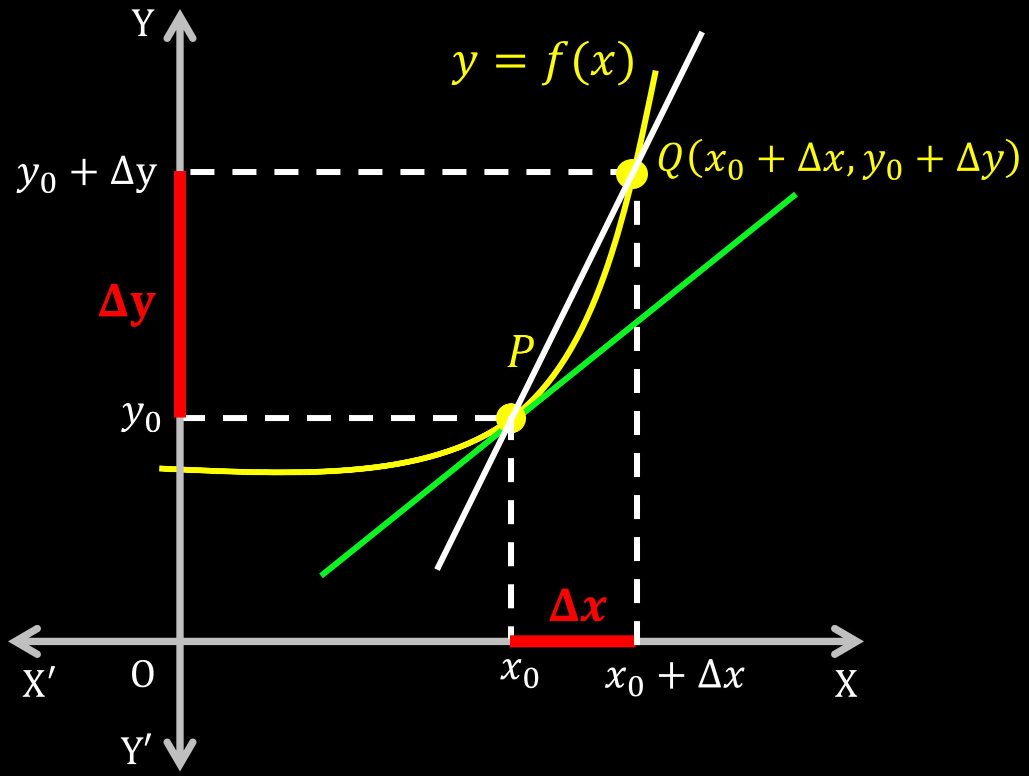
$$\boxed{m_1 \times m_2 = -1}$$

Parallel

Slope $= m_1$

Slope $= m_2$

$$\boxed{m_1 = m_2}$$



$$\text{Slope of } PQ = \frac{(y_0+\Delta y) - y_0}{(x_0+\Delta x) - x_0} = \frac{\Delta y}{\Delta x}$$

$$Q \to P \quad \Rightarrow \Delta x \to 0$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$\text{Slope } m = \left(\frac{dy}{dx}\right)_{(x_0,y_0)}$$

$$\boxed{y - y_0 = m\,(x - x_0)}$$

$$\boxed{y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} (x - x_0)}$$

Ex. Find the equation of the tangent to the curve $\boxed{y = 2x^2 - 1}$ at $\boxed{x = 1}$.

Sol.

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$y_0 = 2(1)^2 - 1 = 1$$

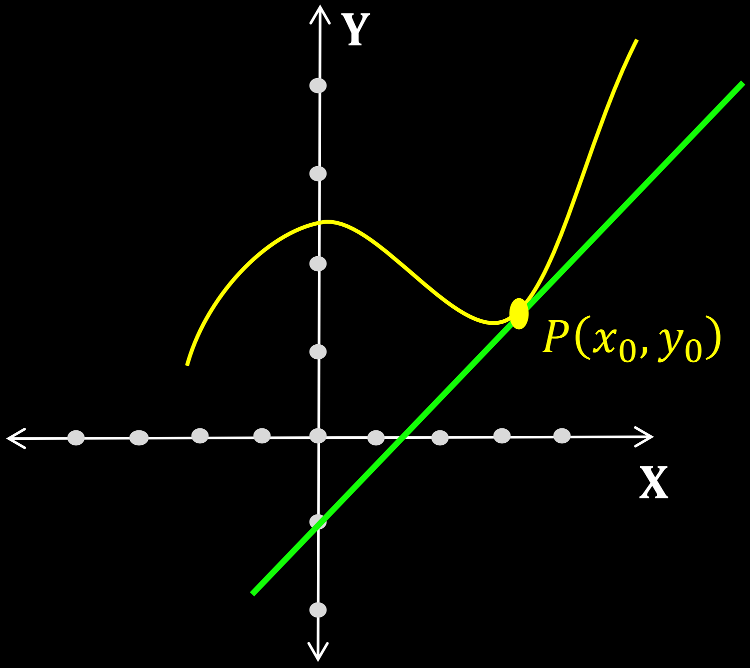
$$\frac{dy}{dx} = 4x$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = 4$$

$$y - 1 = 4(x - 1)$$

$$y - 4x + 3 = 0$$

Q. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$?



$$\frac{dy}{dx} = 3x^2 - 11$$

$$\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = 3x_0^2 - 11$$

$$\boxed{\left(\frac{dy}{dx}\right)_{(x_0, y_0)} = \text{Slope of tangent}}$$

$$\Rightarrow 3x_0^2 - 11 = 1$$

$$\Rightarrow 3x_0^2 = 12 \Rightarrow x_0^2 = 4$$

$$\Rightarrow x_0 = \pm 2$$

$\boxed{x_0 = 2}$ $\boxed{y_0 = (2)^3 - 11(2) + 5 = -9}$

$\boxed{x = -2}$ $\boxed{y_0 = (-2)^3 - 11(-2) + 5 = 19}$

$(2, -9)$ $(-2, 19)$

Q. Find the point on the curve $\boxed{y = x^3 - 11x + 5}$ at which the tangent is $\boxed{y = x - 11}$?

Sol. $\dfrac{dy}{dx} = m$

$$\frac{dy}{dx} = 3x^2 - 11$$

$$\boxed{m = 1}$$

$\Rightarrow 3x^2 - 11 = 1$

$\Rightarrow 3x^2 = 12$

$\Rightarrow x^2 = 4$

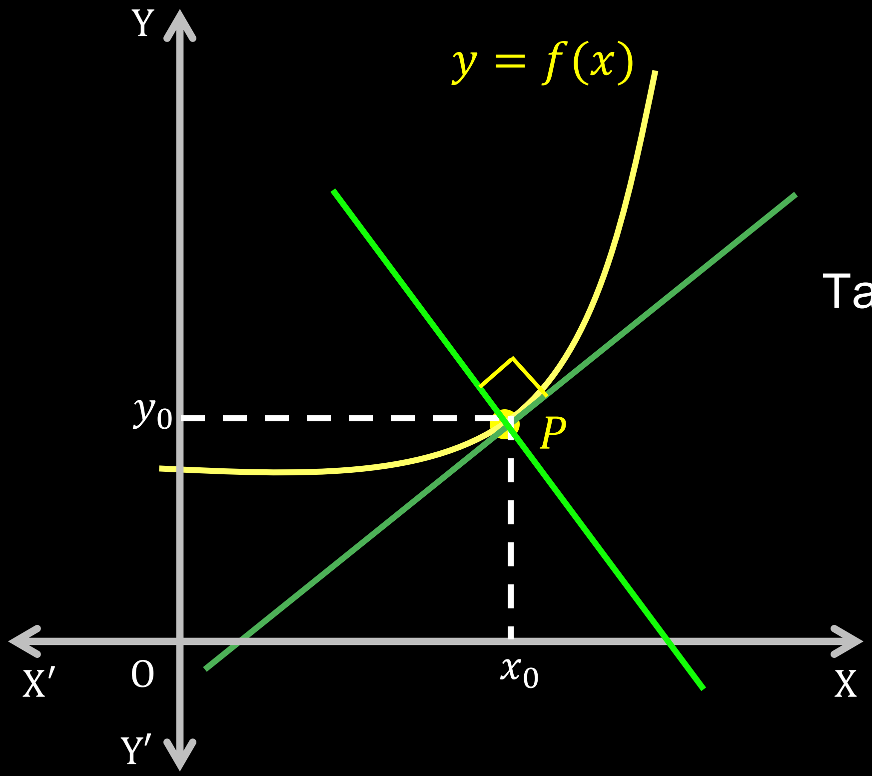
$\Rightarrow x = \pm 2$

$$\boxed{x = 2} \qquad \boxed{x = -2}$$

$$\boxed{y = (2)^3 - 11(2) + 5 = -9} \qquad \boxed{y = (-2)^3 - 11(-2) + 5 = 19}$$

$$\boxed{(2, -9)} \qquad \boxed{(2, 19)}$$

Only point $(2, -9)$ satisfy the given tangent equation.



at point $P(x_0, y_0)$

$$\boxed{m_T = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{(x_0,y_0)}}$$

Tangent $\perp$ Normal

$$\Rightarrow m_T \times m_N = -1$$

$$\Rightarrow m_N = -\frac{1}{m_T}$$

$$\boxed{\Rightarrow m_N = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0,y_0)}}}$$

Ex. Find the slope of the normal to the curve $\boxed{y = x^3 - 3x + 2}$ at $\boxed{x = 3}$.

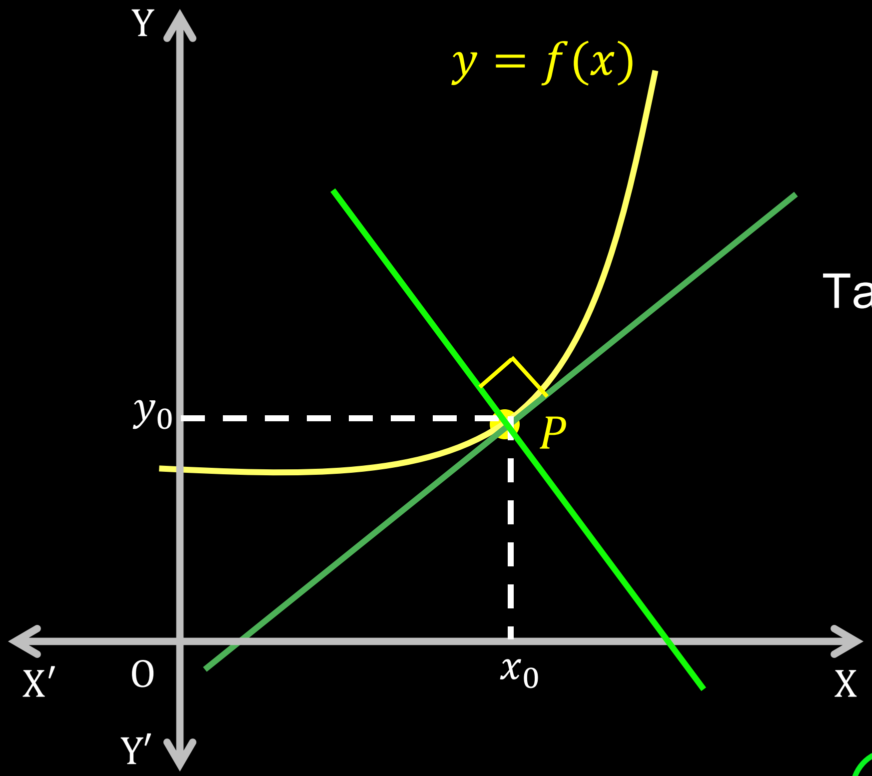
Sol.

Slope $(m_N) = -\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{(x_0, y_0)}}$

$$\frac{dy}{dx} = 3x^2 - 3$$

$$\left(\frac{dy}{dx}\right)_{x=3} = 3(3)^2 - 3 = 24$$

$$= -\frac{1}{24}$$



at point $P(x_0, y_0)$

$$\boxed{m_T = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{(x_0, y_0)}}$$

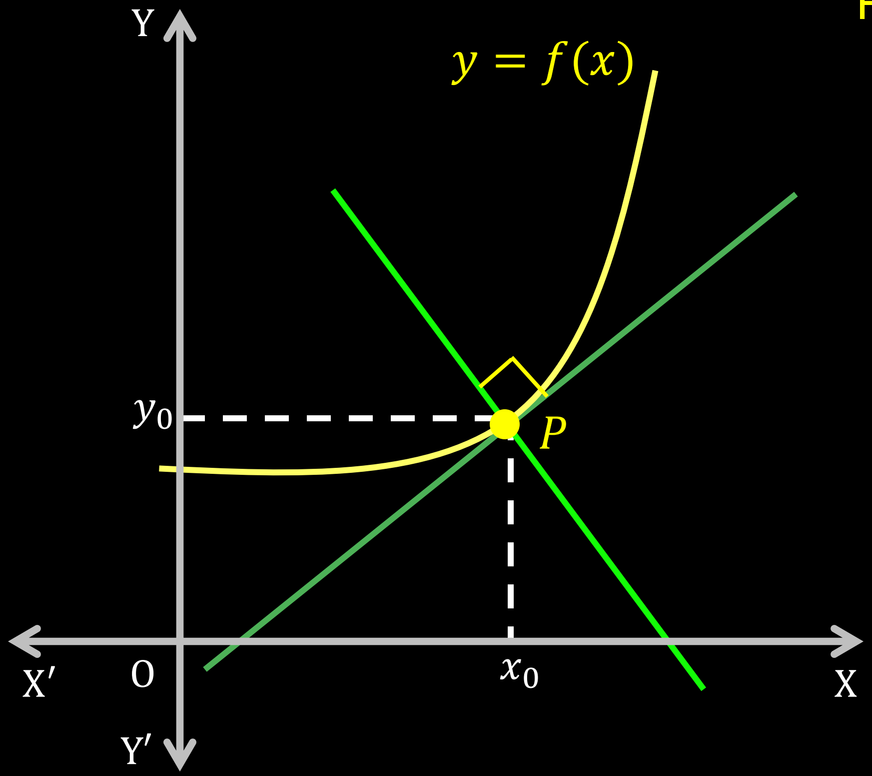
Tangent ⊥ Normal

$$\Rightarrow m_T \times m_N = -1$$

$$\Rightarrow m_N = -\frac{1}{m_T}$$

$$\boxed{\Rightarrow m_N = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}}}$$

$$\boxed{y - y_0 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}}(x - x_0)}$$



Point $P(x_0, y_0)$

Equation of Tangent

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} (x - x_0)$$

Slope of Tangent

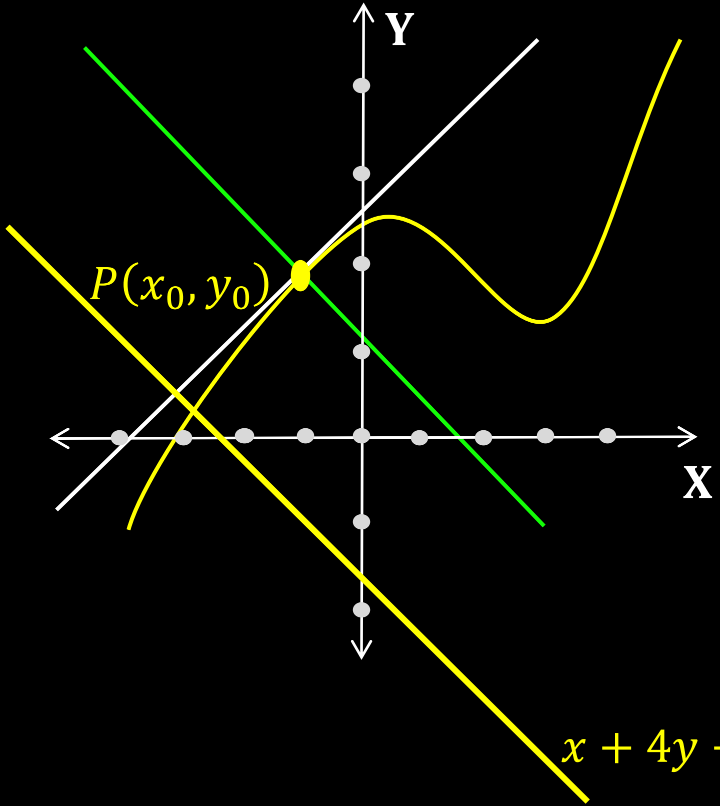
Equation of Normal

$$y - y_0 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_0,y_0)}} (x - x_0)$$

Slope of Normal

Q. Find the equation of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $\boxed{x + 14y + 4 = 0}$.

$$y = -\frac{1}{14}x - \frac{4}{14}$$



$$M_N = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_0,y_0)}}$$

$$\boxed{y - y_0 = m_N(x - x_0)}$$

Q. Find the equation of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

$$m = -\frac{1}{14}$$

$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = 3x_0^2 + 2 = m_T$$

Sol. Equation of normal

$$\boxed{y - y_0 = m_N(x - x_0)}$$

$$\boxed{\Rightarrow m_N = -\frac{1}{3x^2+2}}$$

$$\Rightarrow -\frac{1}{14} = -\frac{1}{3x^2+2}$$

$$\Rightarrow 3x^2 + 2 = 14 \Rightarrow x^2 = 4$$

$$\Rightarrow x = \pm 2$$

$y = x^3 + 2x + 6$	
$x = 2$	$y = 2^3 + 2(2) + 6 = 18$
$x = -2$	$y = (-2)^3 + 2(-2) + 6 = -6$

$(2,18)$

$(-2,-6)$

Equation of normal at point $(2,18)$

$$y - 18 = -\frac{1}{14}(x - 2)$$

$$\Rightarrow \boxed{x + 14y = 254}$$

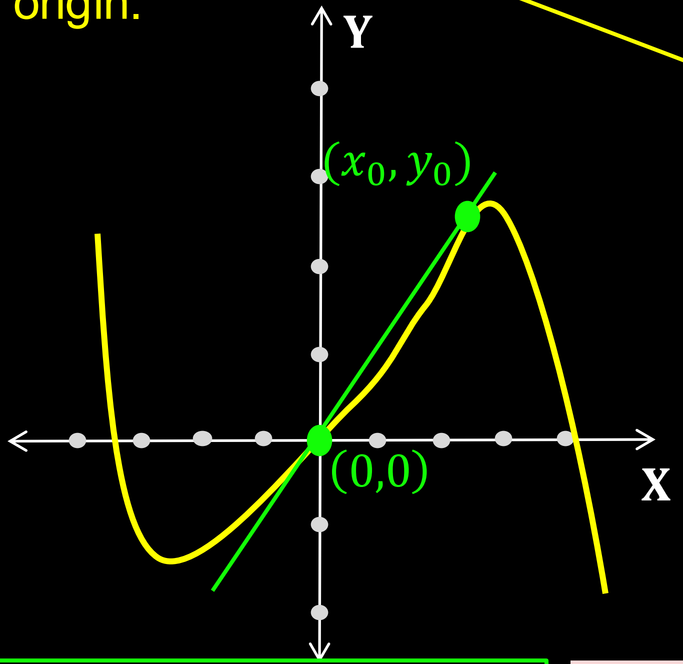
Equation of normal at point $(-2,-6)$

$$y - (-6) = -\frac{1}{14}(x - (-2))$$

$$\Rightarrow \boxed{x + 14y + 86 = 0}$$

Q. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes thought the origin.

Sol.



$$\left(\frac{dy}{dx}\right)_{(x_0,y_0)} = 12x_0^2 - 10x_0^4$$

$$\frac{y_0}{x_0} = 12x_0^2 - 10x_0^4$$

$$\boxed{y_0 = 12x_0^3 - 10x_0^5}$$

$$\boxed{y_0 = 4x_0^3 - 2x_0^5}$$

$$12x_0^3 - 10x_0^5 = 4x_0^3 - 2x_0^5$$

$$\Rightarrow 8x_0^5 - 8x_0^3 = 0$$

$$\Rightarrow 8x_0^3(x_0^2 - 1) = 0$$

$$\Rightarrow x_0 = 0, \pm 1$$

$\boxed{x_0 = -1}$ $\boxed{y_0 = 4(-1)^3 - 2(-1)^5 = -2}$ $(-1, -2)$

$\boxed{x_0 = 1}$ $\boxed{y_0 = 4(1)^3 - 2(1)^5 = 2}$ $(1, 2)$

$\boxed{x_0 = 0}$ $\boxed{y_0 = 0}$ $(0, 0)$

Q. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

Q. Find the equations of the tangent and normal to the hyperbola $\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$ at the point $\boxed{(x_0, y_0)}$.

$$\boxed{\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1}$$

Diff. w.r.t. x

$$\frac{2x}{a^2}-\frac{2y}{b^2}\times\frac{dy}{dx}=0$$

$$\boxed{\Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}}$$

m_T	$m_N=-\frac{1}{m_T}$
$\frac{b^2x_0}{a^2y_0}$	$-\frac{a^2y_0}{b^2x_0}$

Sol. Equation of tangent at point (x_0, y_0)

$$y-y_0=m_T(x-x_0)$$

$$\Rightarrow y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$$

$$\Rightarrow a^2yy_0-a^2y_0^2=b^2xx_0-b^2x_0^2$$

$$\Rightarrow b^2xx_0-a^2yy_0=b^2x_0^2-a^2y_0^2$$

Dividing by a^2b^2

$$\Rightarrow \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}$$

$$\Rightarrow \boxed{\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1}$$

Equation of normal at point (x_0, y_0)

$$y-y_0=m_N(x-x_0)$$

$$\Rightarrow y-y_0=-\frac{a^2y_0}{b^2x_0}(x-x_0) \Rightarrow \frac{y-y_0}{a^2y_0}=-\frac{x-x_0}{b^2x_0} \Rightarrow \boxed{\frac{y-y_0}{a^2y_0}+\frac{x-x_0}{b^2x_0}=0}$$