VIGNAN'S

FOUNDATION FOR SCIENCE, TECHNOLOGY & RESEARCH

(Deemed to be University) - Estd. u/s 3 of UGC Act 1956

Credits: Avanti Sankalp Program

Unit 4: Differential Equations

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Differential Equations

Derivative

$y = 2x^2$

Differentiating w.r.t. x

$\boxed{\frac{dy}{dx} = 4x}$ ✓ Derivative

Equation

$2x + 4 = 8$

$2x + 4y = 25$

$x^2 + 2 = 0$

✗

Differential equation → **Equation** + **Derivative**

$(i)\ \boxed{\frac{dy}{dx}} = 0$ ✓

$(ii)\ \boxed{\frac{d^2y}{dx^2}} + 2 = 0$ ✓

$(iii)\ 5x + 4 = 8$ ✗

Differential Equation (D.E)

Differential Equations

Differential equation → **Equation** + **Derivative**

(i) $\boxed{\dfrac{dy}{dx}} = 0$ ✓

(ii) $\sin\left(\boxed{\dfrac{dy}{dx}}\right) + \cos x = 0$ ✓

(iii) $5x + 4 = 8$ ✗

(iv) $\boxed{\dfrac{d^2u}{dx^2}} + \boxed{\dfrac{d^2u}{dy^2}} = 0$ ✓

Differential Equation (D.E)

Types of D.E

Ordinary Differential Equation

Derivative w.r.t only one independent variable

$$\frac{d^2u}{\boxed{dx^2}} + \frac{du}{\boxed{dx}} = 2x$$

Partial Differential Equation

Derivative w.r.t more than one independent variables

$$\frac{d^2v}{\boxed{dx^2}} + \frac{d^2v}{\boxed{dy^2}} = 0$$ ✗

Notation of Derivative

$$\frac{dy}{dx} \rightarrow \boxed{y'}$$

$$\frac{d^2y}{dx^2} = y''$$

$$\frac{d^3y}{dx^3} = y'''$$

$$\frac{d^{10}y}{dx^{10}} = y''''''''''$$

For Higher Order

$$\frac{d^ny}{dx^n} \rightarrow \boxed{y_n}$$

$$\frac{dy}{dx} = y_1$$

$$\frac{d^8y}{dx^8} = y_8$$

$$\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$\rightarrow \boxed{y''' + 5y'' + 2y' = 0}$$

$$\rightarrow y_3 + 5y_2 + 2y_1 = 0$$

Order of a Differential Equation

Highest derivative present in the D.E

$(i)\ \boxed{\frac{dy}{dx}} = 4x$ Order $= 1$

$(ii)\ \boxed{\frac{d^2y}{dx^2}} + 2\boxed{\frac{dy}{dx}} = 3x$ Order $= 2$

Ex.

$(i)\ \boxed{\frac{d^3y}{dx^3}} + 5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ Order $= 3$

$(ii)\ \frac{dy}{dx} = x\ln\left(\boxed{\frac{d^2y}{dx^2}}\right)$ Order $= 2$

$(iii)\ \left(\boxed{\frac{d^3y}{dx^3}}\right) = x\ln\left(\frac{d^2y}{dx^2}\right)^2$ ~~Order $= 4$~~ Order $= 3$

Note - Order of a differential equation is always a positive integer.

Degree of a Differential Equation

Condition – D.E must be a polynomial equation in derivatives.

Algebraic Equation

Power of variables are whole numbers

$(i)\ 2x - y = 5$ ✓

$(ii)\ \dfrac{3}{x} + y = 6 \Rightarrow 3\boxed{x^{-1}} + y = 6$ ✗

$(iii)\ 2x + \boxed{\sin y} = 0$ ✗

$\dfrac{dy}{dx}$ → Variable

$$(i)\ \left(\boxed{\frac{d^3y}{dx^3}}\right)^1 + 5\left(\boxed{\frac{d^2y}{dx^2}}\right)^2 + 2\left(\boxed{\frac{dy}{dx}}\right)^3 = 0$$ ✓

Order = 3

$$(ii)\ \left(\frac{d^2y}{dx^2}\right)^{\boxed{\frac{1}{2}}} = \left(\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^{\boxed{1}} = \left(\frac{dy}{dx}\right)^{\boxed{2}}$$ ✓

Order = 2

$$(iii)\ \frac{dy}{dx} + sin\left(\boxed{\frac{dy}{dx}}\right) = 0$$ ✗

Order = 1

Degree of a Differential Equation

Highest power of highest order of derivative present in the D.E .

$$(i)\ \left(\boxed{\frac{d^2y}{dx^2}}\right)^{\boxed{2}} + 2\left(\boxed{\frac{dy}{dx}}\right)^{\boxed{4}} = 0$$

Order = 2
Degree = 2

$$(ii)\ \left(\boxed{\frac{d^2y}{dx^2}}\right)^{\boxed{\frac{4}{3}}} = \left(\frac{dy}{dx}\right) + 4$$

Order = 2
Degree = 4

Cubing both sides

$$\left[\left(\frac{d^2y}{dx^2}\right)^{\frac{4}{3}}\right]^3 = \left[\left(\frac{dy}{dx}\right) + 4\right]^3 \Rightarrow \left(\boxed{\frac{d^2y}{dx^2}}\right)^{\boxed{4}} = \left[\left(\frac{dy}{dx}\right) + 4\right]^3$$

Note- Degree of a D.E is always a positive integer.

$(i)\ \boxed{\dfrac{d^4y}{dx^4}} + \sin\left(\boxed{y''}\right) = 0$

Order =

Degree =

$(ii)\ \boxed{y''} + 2y' + \sin y = 0$

Order =

Degree =

$(iii)\ \left(\boxed{\dfrac{d^2y}{dx^2}}\right)^2 + \cos\left(\boxed{\dfrac{dy}{dx}}\right) = 0$

Order =

Degree =

$(iv)\ \left(\dfrac{ds}{dt}\right)^4 + 3s\left(\boxed{\dfrac{d^2s}{dt^2}}\right)^{\boxed{2}} = 0$

Order =

Degree =

$$(i)\ \boxed{\frac{d^4y}{dx^4}} + \sin(\boxed{y''}) = 0$$

Order = 4

Degree = Not defined

$$(ii)\ \boxed{y''} + 2y' + \sin y = 0$$

Order = 2

Degree = 1

$$(iii)\ \left(\boxed{\frac{d^2y}{dx^2}}\right)^2 + \cos\left(\boxed{\frac{dy}{dx}}\right) = 0$$

Order = 2

Degree = Not defined

$$(iv)\ \left(\frac{ds}{dt}\right)^4 + 3s\left(\boxed{\frac{d^2s}{dt^2}}\right)^{\boxed{2}} = 0$$

Order = 2

Degree = 2

Q. Determine order and degree (if defined) of given D. E.

$$\sqrt[5]{\frac{d^2y}{dx^2}+9}=\sqrt{\frac{d^3y}{dx^3}}$$

Sol.

Q. Determine order and degree (if defined) of given D. E.

$$\sqrt[5]{\frac{d^2y}{dx^2} + 9} = \sqrt{\frac{d^3y}{dx^3}}$$

Sol.

$$\left(\frac{d^2y}{dx^2} + 9\right)^{\frac{1}{5}} = \left(\frac{d^3y}{dx^3}\right)^{\frac{1}{2}}$$

Raising both sides to power 10

$$\left(\frac{d^2y}{dx^2} + 9\right)^{2} = \left(\boxed{\frac{d^3y}{dx^3}}\right)^{5}$$

Order $= 3$

Degree $= 5$

Solutions of a Differential Equation

Solution of Equations

- $x^2 - 25 = 0$ $\boxed{x = 5, -5}$

  LHS=RHS

- $x^2 + 36 = 0$ $\boxed{x = \pm 6i}$

Real or Complex Numbers

Solution of D.E

Functions

$$\frac{d\boxed{y}}{dx} - \boxed{y} = 0 \qquad \boxed{\phi}$$

LHS=RHS

ϕ is the solution of D.E

$$\boxed{\frac{dy}{dx} - y = 0}$$

$$\Rightarrow e^x - e^x = 0$$

LHS=RHS

$\boxed{y = e^x}$?

$$\boxed{\frac{dy}{dx} = e^x}$$

Q. Verify that the given function is a solution of corresponding D.E.

$(i)\ y = e^x + 1 \quad : \boxed{y'' - y' = 0}$

Sol. $y = e^x + 1$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(e^x + 1)$$

$\Rightarrow y' = e^x \longrightarrow eq.1$

Differentiating $eq.1$ w.r.t. x

$\Rightarrow y'' = e^x \longrightarrow eq.2$

$y'' - y' = 0$

$\Rightarrow e^x - e^x = 0$

LHS=RHS

$(ii)\ xy = \log y + C \quad : \boxed{y' = \frac{y^2}{1-xy}}$

Sol. $xy = \log y + C$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$$

$\Rightarrow y + xy' = \frac{1}{y} y'$

$\Rightarrow y^2 + xyy' = y'$

$\Rightarrow y^2 = y'(1 - xy)$

$\Rightarrow \boxed{y' = \frac{y^2}{1 - xy}}$

Q. Verify that the given function is a solution of corresponding D.E.

$$x + y = \tan^{-1} y \;:\; \boxed{y^2 y' + y^2 + 1 = 0}$$

Sol.

Q. Verify that the given function is a solution of corresponding D.E.

$$x + y = \tan^{-1} y \;:\; \boxed{y^2 y' + y^2 + 1 = 0}$$

Sol. $x + y = \tan^{-1} y$

Differentiating both sides w.r.t. x

$$\Rightarrow 1 + y' = \left[\frac{1}{1+y^2}\right] y'$$

$$\Rightarrow 1 = \left[\frac{1}{1+y^2} - 1\right] y'$$

$$\Rightarrow 1 = \left[\frac{1-(1+y^2)}{1+y^2}\right] y'$$

$$\Rightarrow 1 = \left[\frac{-y^2}{1+y^2}\right] y'$$

$$\Rightarrow \boxed{y' = \frac{-(1+y^2)}{y^2}}$$

$$\boxed{y^2 y' + y^2 + 1 = 0}$$

$$\Rightarrow y^2\left[\frac{-(1+y^2)}{y^2}\right] + y^2 + 1 = 0$$

$$\Rightarrow -1 - y^2 + y^2 + 1 = 0$$

LHS=RHS

General and Particular Solutions of a D.E

$y = e^x$ $\qquad \dfrac{dy}{dx} - y = 0$

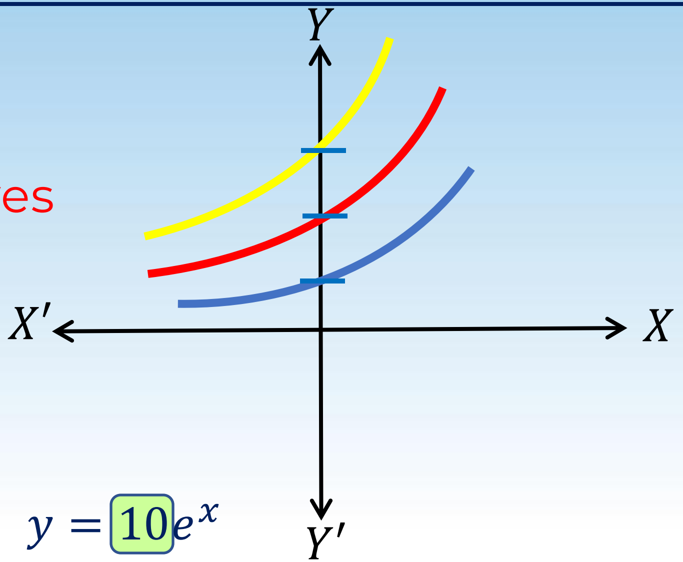
$y = 2e^x$ and $y = 5e^x$

$\Rightarrow \ 2e^x - 2e^x = 0$

$\Rightarrow \ 5e^x - 5e^x = 0$

LHS=RHS

Family of curves



$y = ae^x$,where $a \in R$

$\Downarrow$

Arbitrary Constant

$\Downarrow$

General Solution

$y = 6e^x$ $\qquad y = 10e^x$

Specific Values

$\Downarrow$

Particular Solution

Summary

➢ Differential Equation

- Equation involving derivative of dependent variable w.r.t. independent variable.

➢ Order of Differential Equation

- Highest order of derivative present in the Differential Equation

➢ Degree of Differential Equation

- D.E must be a polynomial equation in derivatives
- Highest power of highest order of derivative present in the D.E .

➢ General and Particular Solutions of a differential equation

- A function is obtained in the solution of Differential Equation .

❑ General solution contains arbitrary constants

❑ Particular solution contains no arbitrary constants but only the particular values

➢ Formation of a differential equation

❖ Order of D.E is equal to the number of arbitrary constants in general solution..

Special Cases to solve DEs

- Variable-separable method
- Homogeneous Equations
- Linear Equations

Variables Separable type D.E.

First Order, First Degree D.E.

✓ $$\frac{dy}{dx} = \frac{x+1}{y+2} \Rightarrow \boxed{(x+1)dx} = \boxed{(y+2)dy}$$

✗ $$\frac{d^2y}{dx^2} = \frac{y}{x}$$

✓ $$\frac{dy}{dx} + y = 1 \Rightarrow \frac{dy}{dx} = \boxed{1-y} \Rightarrow \frac{1}{(1-y)}dy = dx$$

✓ $$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

$$\Rightarrow \sec^2 x \tan y \, dx = -\sec^2 y \tan x \, dy$$

$$\Rightarrow \boxed{\frac{\sec^2 x}{\tan x} dx} = \boxed{-\frac{\sec^2 y}{\tan y} dy}$$

Variables Separable type D.E

✓ $$\frac{dy}{dx} = e^{x+y}$$

$$\Rightarrow \frac{dy}{dx} = e^x . e^y$$

$$\Rightarrow \boxed{\left(\frac{1}{e^y}\right) dy} = \boxed{(e^x)dx}$$

Variables Separable Method

General Solution ➡ D.E.

Differentiation

$y = \boxed{m}x$

$\dfrac{dy}{dx} = m$

$\dfrac{dy}{dx} = \dfrac{y}{x}$

Integration

D.E. ➡ General and Particular Solution

General Solution

- Separate the variables
- Integrate both sides

Particular Solution

- Eliminate arbitrary constant from G.S. using given information

$$\frac{dy}{dx} = -7x \quad ; \quad y = 3 \text{ when } x = 0$$

$$dy = -7x\,dx$$

$$\int dy = \int -7x\,dx$$

$$y + C_1 = -\frac{7}{2}x^2 + C_2$$

$$y = -\frac{7}{2}x^2 + C_2 - C_1$$

$$\boxed{y = -\frac{7}{2}x^2 + C}$$

General Solution

$$3 = -\frac{7}{2}(0)^2 + C$$

$$\boxed{C = 3}$$

$$\boxed{y = -\frac{7}{2}x^2 + 3}$$

Particular Solution

Q. For differential equation find a particular solution satisfying the given condition .

$$\cos\left(\frac{dy}{dx}\right) = a\ (a \in R); y = 1 \text{ when } x = 0$$

Sol.

Q. For differential equation find a particular solution satisfying the given condition .

$$\cos\left(\frac{dy}{dx}\right) = a\ (a \in R); y = 1 \text{ when } x = 0$$

Sol. Separating the variables

$$\frac{dy}{dx} = \cos^{-1} a$$

$$\Rightarrow dy = (\cos^{-1} a)dx$$

Integrating both sides

$$\int dy = \cos^{-1} a \int dx$$

$$\Rightarrow y = \cos^{-1} a\ (x) + C$$

$$\Rightarrow \boxed{y = x\cos^{-1} a + C} \longrightarrow Eq.1$$

General Solution

$y = 1$ when $x = 0$

$$\Rightarrow 1 = 0 \times \cos^{-1} a + C$$

$$\Rightarrow C = 1$$

Substituting $C = 1$ in $Eq.1$

$$\Rightarrow y = x\cos^{-1} a + 1$$

$$\Rightarrow \frac{y-1}{x} = \cos^{-1} a$$

$$\Rightarrow \boxed{\cos\left(\frac{y-1}{x}\right) = a}$$

Required Particular Solution

Q. Find the equation of curve passing through the point $(0, -2)$ given that at any point (x, y) on the curve ,the product of the slope of its tangent and y −coordinate of the point is equal to the x − coordinates of the point.

Sol.

Q. Find the equation of curve passing through the point $(0, -2)$ given that at any point (x, y) on the curve ,the product of the slope of its tangent and y –coordinate of the point is equal to the x – coordinates of the point.

Sol. Slope of tangent to the curve = $\frac{dy}{dx}$

$$y\frac{dy}{dx} = x$$

Separating the variables

$$\Rightarrow y\,dy = x\,dx$$

Integrating both sides

$$\int y\,dy = \int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\Rightarrow \boxed{y^2 - x^2 = 2C} \longrightarrow Eq.1$$

General Solution

Curve passes through point $(0, -2)$

$$\Rightarrow (-2)^2 - (0)^2 = 2C$$

$$\Rightarrow 2C = 4$$

Substituting $2C = 4$ in $Eq.1$

$$\Rightarrow \boxed{y^2 - x^2 = 4}$$

Particular Solution

Required Equation

Summary

- **Variable Separable Differential Method**

$$\frac{dy}{dx} = f(x, y)$$ (Variables Separable Type D.E.)

- General Solution
  - Separate the variables
  - Integrate both sides

- Particular Solution
  - Find the General solution
  - Eliminate arbitrary constant from General Solution using given information in question

Homogenous Differential Equations

$$\frac{dy}{dx} = \frac{x+3y}{x-y}$$ ✗

Separate the variables

Homogenous Differential Equations

Homogenous Function ➡ Multiplicative Scaling Behaviour

Arguments $\xrightarrow{\textit{Multiplied by}}$ Factor

Value of function $\xrightarrow{\textit{Multiplied by}}$ Power of that Factor

⬇

Degree

➢ $F(x,y) = \boxed{5x^2 + 4y^2}$ ✓ $\boxed{\lambda}$

$x = \lambda x$ and $y = \lambda y$

$\Rightarrow F(\lambda x, \lambda y) = 5(\lambda x)^2 + 4(\lambda y)^2$

$= \lambda^2 \boxed{(5x^2 + 4y^2)}$

$\Rightarrow F(\lambda x, \lambda y) = \lambda^{\boxed{2}} F(x,y)$

Degree $= 2$

➢ $F(x,y) = \boxed{\sin\left(\frac{y}{x}\right)}$ ✓

$y = \lambda y$ and $x = \lambda x$

$\Rightarrow F(\lambda x, \lambda y) = \sin\left(\frac{\lambda y}{\lambda x}\right)$

$= \lambda^0 \boxed{\sin\left(\frac{y}{x}\right)}$

$\Rightarrow F(\lambda x, \lambda y) = \lambda^{\boxed{0}} F(x,y)$

Degree $= 0$

➢ $F(x,y) = \cos x + \sin y$ ✗

$y = \lambda y$ and $x = \lambda x$

$\Rightarrow F(\lambda x, \lambda y) = \cos \lambda x + \sin \lambda y$

$\Rightarrow F(\lambda x, \lambda y) = \lambda^n F(x,y)$

Not Possible

Homogenous Differential Equations

Alternate Method

- $F(x,y) = x^n g\left(\frac{y}{x}\right)$
- $F(x,y) = y^n h\left(\frac{x}{y}\right)$

Homogenous Function

Degree $= n$

- $F(x,y) = \cos x + \sin y$
  $= x^n\ g\left(\frac{y}{x}\right)$ ✗
  Or
  $= y^n\ h\left(\frac{x}{y}\right)$ ✗

$$\frac{dy}{dx} = \boxed{F(x,y)} \Rightarrow \text{Homogenous Function}$$

Degree $= 0$

Homogenous D.E.

Homogenous Differential Equations

Alternate Method

- $F(x,y) = x^n g\left(\frac{y}{x}\right)$
- $F(x,y) = y^n h\left(\frac{x}{y}\right)$

Homogenous Function

Degree $= n$

- $F(x,y) = 5x + 4y$

$$= x\left(5 + \frac{4y}{x}\right) = x^1 g\left(\frac{y}{x}\right)$$

Or

$$= y\left(\frac{5x}{y} + 4\right) = y^1 g\left(\frac{x}{y}\right)$$

Degree $= 1$

- $F(x,y) = \sin\left(\frac{y}{x}\right)$

$$= x^0 \sin\left(\frac{y}{x}\right) = x^0 h\left(\frac{y}{x}\right)$$

Or

$$= y^0 \sin\left(\frac{x}{y}\right) = y^0 h\left(\frac{x}{y}\right)$$

Degree $= 0$

- $F(x,y) = \cos x + \sin y$

$$= x^n\, g\left(\frac{y}{x}\right) \times$$

Or

$$= y^n\, h\left(\frac{x}{y}\right) \times$$

$$\frac{dy}{dx} = \boxed{F(x,y)} \Rightarrow$$

Homogenous Function

Degree $= 0$

Homogenous D.E.

Homogenous Differential Equations

Ex. Show that the $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ is a homogenous differential equation

Sol.

Method I

$$F(x,y) = \frac{x+y}{x-y}$$

$$F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y}$$

$$= \frac{\lambda(x+y)}{\lambda(x-y)}$$

$$= \lambda^0 F(x,y)$$

Degree $= \mathbf{0}$

Method II

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$$= \frac{x\left(1+\frac{y}{x}\right)}{x\left(1-\frac{y}{x}\right)}$$

$$= x^0\, g\left(\frac{y}{x}\right)$$

Degree $= \mathbf{0}$

$$= \frac{y\left(\frac{x}{y}+1\right)}{y\left(\frac{x}{y}-1\right)}$$

$$= y^0\, g\left(\frac{x}{y}\right)$$

Q. Find the particular solution of given differential equation satisfying the given condition

$$x^2dy + (xy + y^2)dx = 0\,;\, y = 1 \text{ when } x = 1$$

Sol.

Q. Find the particular solution of given differential equation satisfying the given condition

$$x^2dy + (xy + y^2)dx = 0\,;\, y = 1 \text{ when } x = 1$$

Sol.

$$x^2dy + (xy + y^2)dx = 0$$

$$\frac{dy}{dx} = \frac{-(xy+y^2)}{x^2}$$

$$F(x, y) = \frac{-(xy+y^2)}{x^2}$$

$$F(\lambda x, \lambda y) = \frac{-(\lambda x\lambda y+(\lambda y)^2)}{(\lambda x)^2}$$

$$F(\lambda x, \lambda y) = \lambda^0 F(x, y)$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting value of y and $\frac{dy}{dx}$ in given D.E.

$$v + x\frac{dv}{dx} = \frac{-(x.vx+(vx)^2)}{x^2}$$

$$\frac{dv}{v(v+2)} = -\frac{dx}{x}$$

$$\int\frac{dv}{v(v+2)} = -\int\frac{dx}{x}$$

$$\frac{1}{2}\left[\frac{(v+2)-v}{v(v+2)}\right] = -\frac{dx}{x}$$

$$\frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right] = -\frac{dx}{x}$$

$$\frac{1}{2}[\log v - \log(v+2)] = -\log x + \log C$$

$$\frac{1}{2}\log\left(\frac{v}{v+2}\right) = \log\left(\frac{C}{x}\right)$$

$$\left(\frac{v}{v+2}\right) = \left(\frac{C}{x}\right)^2$$

$$\left(\frac{\frac{y}{x}}{\frac{y}{x}+2}\right) = \left(\frac{C}{x}\right)^2$$

$$\left(\frac{x^2y}{y+2x}\right) = C^2$$

$$C^2 = \frac{1}{3}$$

$$\frac{x^2y}{y+2x} = \frac{1}{3}$$

$$y + 2x = 3x^2y$$

Summary

- **Homogenous Differential Equation**

(Variables Separable Type D.E)

- General Solution
  - Separate the variables
  - Integrate both sides

- Particular Solution
  - Find the General solution
  - Eliminate arbitrary constant from General Solution using given information in question

Linear Differential Equations

$$x\frac{dy}{dx} + 2y = x^2$$ ✗

- ❑ Homogenous D.E.
- ❑ Variable Separable Method

Linear Equation

Highest power of variable is 1 .

✓ $2x + 3 = 0$	1
✓ $x + y = 2$	1
✗ $x^2 + 1 = 0$	2

Dependent variable and its Derivative ⇒ 1^{st} Degree

⇓

Not multiplied together

$$\boxed{\frac{dy}{dx} + Py = Q}$$

Constant or Function of x

Linear D.E. in variable y

$$\boxed{\frac{dx}{dy} + Px = Q}$$

Constant or Function of y

Linear D.E. in variable x

$$x\frac{dy}{dx} + 2y = x^2$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x$$

$$\frac{dx}{dy} + \frac{2}{y}x = 5$$

Q. Check whether the given differential equations are linear differential equations or not.

✗ ➢ $dy = (5 - 2y^2)dx$ $\Rightarrow \boxed{\frac{dy}{dx}} + 2\boxed{y^2} = 5$

✓ ➢ $\boxed{\frac{d^2x}{dy^2}} - \boxed{x} = 5y$

Order= 2

Degree= 1

✗ ➢ $\frac{dy}{dx} + \frac{2}{y} = 5$ $\Rightarrow \frac{dy}{dx} + 2\boxed{y^{-1}} = 5$

Solving Linear Differential Equations

Integrable Form

$$\frac{dy}{dx} + Py = Q$$

× Solution ?

Integrating Factor

(Special Function)

$$\frac{dx}{dy} + Px = Q$$

Linear D.E. in variable y

Linear D.E. in variable x

$f(x)$

$$f(x)\frac{dy}{dx} + Pf(x)y = Qf(x)$$

$f(y)$

$$\frac{d}{dx}(yf(x)) = f(x)\frac{dy}{dx} + yf'(x)$$

$$f(x) = e^{\int P dx} = \text{I.F.}$$

$$f(y) = e^{\int P dy} = \text{I.F.}$$

$$Pf(x) = f'(x)$$

$$P = \frac{f'(x)}{f(x)}$$

Integrating w.r.t x

$$\int P dx = \int \frac{f'(x)}{f(x)} dx$$

$$\int P dx = \log|f(x)|$$

$$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx}y = Qe^{\int P dx}$$

$$\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

Integrating w.r.t x

Solution

$$ye^{\int P dx} = \int \left(Qe^{\int P dx}\right)dx + C$$

$$xe^{\int P dy} = \int \left(Q \times e^{\int P dy}\right)dy + C$$

× Solution

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})dx + C$$

$$x \times \text{I.F.} = \int (Q \times \text{I.F.})dy + C$$

Solving Linear Differential Equations

Linear D.E. in variable y

$$\frac{dy}{dx} + Py = Q \quad \times \quad f(x)$$

$$(yf(x))' = y'f(x) + yf'(x)$$

Solution?

$$f(x)\frac{dy}{dx} + Pf(x)y = Qf(x)$$

$$(yf(x))' = Qf(x)$$

Integrating w.r.t. x

$$\int (yf(x))' dx = \int Qf(x)dx$$

$$y \times f(x) = \int Qf(x)dx$$

$$y = \frac{1}{f(x)}\int Qf(x)dx$$

$$Pf(x) = f'(x)$$

$$P = \frac{f'(x)}{f(x)}$$

Integrating w.r.t. x

$$\int Pdx = \log f(x)$$

$$f(x) = e^{\int Pdx} = \text{I.F.}$$

$$ye^{\int Pdx} = \int \left(Qe^{\int Pdx}\right)dx + C$$

$$y \times \text{I.F.} = \int (Q \times \text{I.F.})dx + C$$

$$\frac{dx}{dy} + Px = Q$$

Linear D.E. in variable x

$$f(y)$$

Integrating Factor

$$f(y) = e^{\int Pdy} = \text{I.F.}$$

$$xe^{\int Pdy} = \int \left(Q \times e^{\int Pdy}\right)dy + C$$

$$x \times \text{I.F.} = \int (Q \times \text{I.F.})dy + C$$

Solving Linear Differential Equations

Steps

1. Standard form
2. $\boxed{\frac{dy}{dx} + \boxed{P}y = \boxed{Q}}$ Form
3. Identify P and Q
4. Find $\mathrm{I.F.} = e^{\int P dx}$
5. $y(\mathrm{I.F.}) = \int (Q \times \mathrm{I.F})dx + C$

$[e^{\log f(x)} = f(x)]$

$$\frac{dy}{dx} = x^2 - \frac{y}{x}$$

General Solution ?

$$\boxed{\frac{dy}{dx}} + \boxed{\frac{1}{x}}\boxed{y} = \boxed{x^2}$$

$\boxed{P = \frac{1}{x}}$ $\boxed{Q = x^2}$

$$\boxed{\frac{dx}{dy} + Px = Q}$$

$\boxed{\mathrm{I.F.} = e^{\int P dy}}$

$\boxed{x(\mathrm{I.F}) = \int (Q \times \mathrm{I.F.})dy + C}$

$$\mathrm{I.F.} = e^{\int P dx}$$

$$\mathrm{I.F.} = e^{\boxed{\int \frac{1}{x} dx}}$$

$$\mathrm{I.F.} = e^{\log x}$$

$$\boxed{\mathrm{I.F.} = x}$$

$$y(\mathrm{I.F.}) = \int (Q \times \mathrm{I.F.})dx + C$$

$$yx = \int (x^2(x))dx + C$$

$$xy = \int (x^3)dx + C$$

$$\boxed{xy = \frac{x^4}{4} + C}$$

Solution

Q. Find the general solution for given differential equation

$$\boxed{\frac{dy}{dx} + 2y = \sin x}$$

Sol.

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$\boxed{P = 2}$ and $\boxed{Q = \sin x}$

$$\text{I.F.} = e^{\int P dx}$$

$$\text{I.F.} = e^{\int 2 dx} = \boxed{e^{2x}}$$

Solution

$$y(\text{I.F.}) = \int (Q \times \text{I.F.}) dx + C$$

$$ye^{2x} = \boxed{\int \sin x \, e^{2x} dx} + C$$

$$I = \int \sin x \, e^{2x} dx$$

$$I = \sin x \int e^{2x} dx - \int \left(\frac{d}{dx}(\sin x) \int e^{2x} dx \right) dx$$

$$I = \frac{e^{2x} \sin x}{2} - \boxed{\int \left(cosx \frac{e^{2x}}{2} \right) dx}$$

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos x \int e^{2x} - \int \left(\frac{d}{dx}(\cos x) \int e^{2x} dx \right) dx \right]$$

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left[\cos x \frac{e^{2x}}{2} - \int \left((-\sin x) \frac{e^{2x}}{2} \right) dx \right]$$

$$I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} \boxed{\int (\sin x \, e^{2x}) dx}$$

$$I = \frac{e^{2x}}{4} (2 \sin x - \cos x) - \frac{1}{4} I$$

$$\boxed{I = \frac{e^{2x}}{5} (2 \sin x - \cos x)}$$

$$ye^{2x} = \frac{e^{2x}}{5} (2 \sin x - \cos x) + C$$

$$\boxed{y = \frac{1}{5} (2 \sin x - \cos x) + Ce^{-2x}}$$

G.S.

Q. Find the general solution for given differential equation

$$\boxed{(x+y)\frac{dy}{dx}=1}$$

Sol.

Q. Find the general solution for given differential equation

$$(x+y)\frac{dy}{dx} = 1$$

Sol. $\boxed{\frac{dy}{dx}} = \frac{1}{(x+y)}$ ✗

$$\Rightarrow \frac{dx}{dy} = x + y \qquad \Rightarrow \frac{dx}{dy} - x = y$$

$$\boxed{\frac{dx}{dy} + Px = Q}$$

$\boxed{P = -1}$ and $\boxed{Q = y}$

$$\text{I. F.} = e^{\int P dy}$$

$$\text{I. F.} = e^{\int -1 dy} = \boxed{e^{-y}}$$

$$\boxed{x(\text{I. F.}) = \int (Q \times \text{I. F.}) dy + C}$$

$$xe^{-y} = \int (ye^{-y}) dy + C$$

$$xe^{-y} = y\int (e^{-y}) dy - \int \left[\frac{d}{dy}(y)\int e^{-y} dy\right] + C$$

$$xe^{-y} = y(-e^{-y}) dy + \boxed{\int (e^{-y}) dy + C}$$

$$xe^{-y} = -ye^{-y} - e^{-y} + C$$

$$x = -y - 1 + Ce^{y}$$

$$\boxed{x + y + 1 = Ce^{y}}$$ **G.S.**

Q. Find the equation of curve passing through the point $(0,2)$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at any point by 5.

Sol.

Q. Find the equation of curve passing through the point $(0,2)$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at any point by 5 .

Sol. $F(x,y)$

(x,y) Slope $= \frac{dy}{dx}$

$$\frac{dy}{dx} + 5 = x + y \quad \Rightarrow \frac{dy}{dx} - y = x - 5$$

$$\frac{dy}{dx} + Py = Q$$

$\boxed{P = -1}$ and $\boxed{Q = x - 5}$

$$\mathrm{I.F.} = e^{\int P dx}$$

$$\mathrm{I.F.} = e^{\int (-1) dx} = e^{-x}$$

$$y(\mathrm{I.F.}) = \int (Q \times \mathrm{I.F.}) dx + C$$

$$ye^{-x} = \int (x-5)e^{-x} dx + C$$

$$\int (x-5)e^{-x}dx = (x-5)\int e^{-x}dx - \int \left[\frac{d}{dx}(x-5)\int e^{-x}dx\right] dx$$

$$= (x-5)(-e^{-x}) - \int (-e^{-x}) dx$$

$$= (5-x)e^{-x} - e^{-x}$$

$$\int (x-5)e^{-x}dx = (4-x)e^{-x}$$

$$ye^{-x} = (4-x)e^{-x} + C$$

$$y = 4 - x + Ce^{x}$$

$$x + y - 4 = Ce^{x}$$ G.S.

$\boxed{x = 0}$

$\boxed{y = 2}$

$$C = -2$$

$$x + y - 4 = -2e^{x}$$ Required Equation

Summary

- **Linear Differential Equation**

  Dependent variable and its Derivative are of 1^{st} Degree and are not multiplied together.

- **Standard Forms**
  - $\frac{dy}{dx} + Py = Q$
  - $\frac{dx}{dy} + Px = Q$

- **Solving Homogenous Differential Equation**
  1. Write D.E. in Standard form
  2. Identify the type
  3. Identify P and Q
  4. Find $\mathrm{I.F.}\ e^{\int Pdx}$ or $e^{\int Pdy}$ accordingly
  5. Find $y(\mathrm{I.F.}) = \int (Q \times \mathrm{I.F.})dx + C$

     or

     $x \times \mathrm{I.F} = \int (Q \times \mathrm{I.F.})dy + C$ accordingly