SCHOOL OF APPLIED SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

Subject: Linear Algebra | Subject Code : 25MT103

Sem. : I | Academic Year: 2025-2026

Section: 7 | Regulation: R25

Module Bank

Question 1 (10 Marks)

a) Define the characteristic equation of a matrix. Explain how eigenvalues are obtained from the characteristic equation. **(2 marks)**

b) Find the characteristic equation and eigenvalues of the matrix A = $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Verify your eigenvalues by substituting back into the characteristic equation. **(4 marks)**

c) Check that the sum of eigenvalues of a matrix equals its trace, and the product of eigenvalues equals its determinant. Demonstrate this property with a 3×3 matrix of your choice. **(4 marks)**

Question 2 (10 Marks)

a) State the Cayley-Hamilton Theorem. Indicate whether the theorem holds for every n × n matrix over the complex numbers. **(2 marks)**

b) Verify the Cayley-Hamilton Theorem for the matrix A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Use this result to compute A^3. **(4 marks)**

c) Outline the Cayley-Hamilton recipe that rewrites A^n as a linear combination of I and integer powers of A. Demonstrate with a specific example for n = 3. **(4 marks)**

Question 3 (10 Marks)

a) Define a diagonalizable matrix. State the necessary and sufficient condition for a matrix to be diagonalizable. **(2 marks)**

b) Determine whether the matrix A =$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is diagonalizable. If yes, find matrices P and D such that $A = PDP^{-1}$. If no, justify your answer. **(4 marks)**

c) Give one concrete 3×3 matrix that has three distinct eigenvalues and show that it is diagonalizable. Construct a counterexample showing that a matrix can be diagonalizable even without *n* distinct eigenvalues. **(4 marks)**

Question 4 (10 Marks)

a) Define a real vector space and list the eight axioms that must be satisfied by the addition and scalar multiplication operations. **(3 marks)**

b) Verify whether the set $V = \{(x, y) : x, y \in \mathbb{R}, x \geq 0\}$ with standard addition and scalar multiplication forms a vector space. **(3 marks)**

c) Can you give examples of vector space each with different operations but the same underlying set, where one is vector space and the other is not. **(4 marks)**

Question 5 (10 Marks)

a) Explain the concepts of linear combination and linear span of a set of vectors. **(2 marks)**

b) Express the vector $v = (2, -1, 5)$ as a linear combination of $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 2)$, and $u_3 = (1, 1, 0)$. Also, determine $\text{span}\{u_1, u_2, u_3\}$. **(4 marks)**

c) Show that $\text{span}\{v_1, v_2, ..., v_n\}$ is the smallest subspace containing all the vectors $v_1, v_2, ..., v_n$. Create a geometric interpretation for $n = 2$ in $\mathbb{R}^3$. **(4 marks)**

Question 6 (10 Marks)

a) Define the row space and column space of an m×n matrix A. How are they related to the matrix operations? **(3 marks)**

b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$, find: (i) A basis for the row space (ii) A basis for the column space (iii) Verify that $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$ **(4 marks)**

c) Explain why elementary row operations preserve the row space but may change the column space. **(3 marks)**

Question 7 (10 Marks)

a) What is the difference between an inner product and a norm? How are they related? **(3 marks)**

b) Let $u = (1,2)$and $v = (2,3)$. Find the inner product u and v. Also, find angle between u and v using the inner product. **(3 marks)**

c) Analyze whether the following function is an inner product on $\mathbb{R}^2$: $\langle(x_1, x_2), (y_1, y_2)\rangle = 3x_1y_1 + x_2y_2$. Justify using axioms. **(4 marks)**

Question 8 (10 Marks)

a) Define the norm of a vector in an inner product space and express it in terms of the inner product. **(2 marks)**

b) For the vectors $u = (1, 2, 3)$ and $v = (4, 5, 6)$ in R^3 with the standard inner product, calculate $||u||$, $||v||$, and $||u + v||$. **(4 marks)**

c) Prove that for any vector v in an inner product space, $||\alpha v|| = |\alpha|\ ||v||$ where α is a scalar. Discuss the geometric significance of this property. **(4 marks)**

Question 9 (10 Marks)

a) Recall the relationship between SVD and the four fundamental subspaces of a matrix. **(2 marks)**

b) Given a matrix A with SVD $A = U\Sigma V^{T}$ where $\Sigma = diag(5, 3, 0)$, determine the rank of A, and identify the dimensions of its null space and column space. **(4 marks)**

c) Design a procedure to use SVD for image compression. Analyze how the number of singular values retained affects image quality and storage requirements. Provide a mathematical framework for your analysis. **(4 marks)**

Question 10 (10 Marks)

a) State the theorem regarding the uniqueness of QR decomposition. Under what conditions is the decomposition unique? **(2 marks)**

b) Given the matrix $A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 1 \end{pmatrix}$, perform QR decomposition and verify your answer by computing QR. **(5 marks)**

c) Construct an algorithm to use QR decomposition iteratively to find eigenvalues of a matrix (QR algorithm). Explain the convergence properties. **(3 marks)**