Linear Algebra -25MT103 - Module Bank

Section: 7, 14, 21 **Faculty:** Dr. D Bhanu Prakash

Question 1 (10 Marks)

a) Define the characteristic equation of a matrix. Explain how eigenvalues are obtained from the characteristic equation. **(2 marks)**

b) Find the characteristic equation and eigenvalues of the matrix A = $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$. Verify your eigenvalues by substituting back into the characteristic equation. **(4 marks)**

c) Check that the sum of eigenvalues of a matrix equals its trace, and the product of eigenvalues equals its determinant. Demonstrate this property with a 3×3 matrix of your choice. **(4 marks)**

Question 2 (10 Marks)

a) State the Cayley-Hamilton Theorem. Indicate whether the theorem holds for every n × n matrix over the complex numbers. **(2 marks)**

b) Verify the Cayley-Hamilton Theorem for the matrix A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Use this result to compute A^3. **(4 marks)**

c) Outline the Cayley-Hamilton recipe that rewrites A^n as a linear combination of I and integer powers of A. Demonstrate with a specific example for n = 3. **(4 marks)**

Question 3 (10 Marks)

a) Define a diagonalizable matrix. State the necessary and sufficient condition for a matrix to be diagonalizable. **(2 marks)**

b) Determine whether the matrix A = $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is diagonalizable. If yes, find matrices P and D such that $A = PDP^{-1}$. If no, justify your answer. **(4 marks)**

c) Give one concrete 3×3 matrix that has three distinct eigenvalues and show that it is diagonalizable. Construct a counterexample showing that a matrix can be diagonalizable even without *n* distinct eigenvalues. **(4 marks)**

Question 4 (10 Marks)

a) Define a real vector space and list the eight axioms that must be satisfied by the addition and scalar multiplication operations. **(3 marks)**

b) Verify whether the set $V = \{(x, y) : x, y \in \mathbb{R}, x \geq 0\}$ with standard addition and scalar multiplication forms a vector space. **(3 marks)**

c) Can you give examples of vector space each with different operations but the same underlying set, where one is vector space and the other is not. **(4 marks)**

Question 5 (10 Marks)

a) Explain the concepts of linear combination and linear span of a set of vectors. **(2 marks)**

b) Express the vector $v = (2, -1, 5)$ as a linear combination of $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 2)$, and $u_3 = (1, 1, 0)$. Also, determine $\text{span}\{u_1, u_2, u_3\}$. **(4 marks)**

c) Show that $\text{span}\{v_1, v_2, ..., v_n\}$ is the smallest subspace containing all the vectors $v_1, v_2, ..., v_n$. Create a geometric interpretation for $n = 2$ in $\mathbb{R}^3$. **(4 marks)**

Question 6 (10 Marks)

a) Define the row space and column space of an m×n matrix A. How are they related to the matrix operations? **(3 marks)**

b) For the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 2 \end{pmatrix}$, find: (i) A basis for the row space (ii) A basis for the column space (iii) Verify that dim(Row(A)) = dim(Col(A)) **(4 marks)**

c) Explain why elementary row operations preserve the row space but may change the column space. **(3 marks)**

Question 7 (10 Marks)

a) What is the difference between an inner product and a norm? How are they related? **(3 marks)**
b) Let $u = (1,2)$ and $v = (2,3)$. Find the inner product u and v. Also, find angle between u and v using the inner product. **(3 marks)**
c) Analyze whether the following function is an inner product on R^2: $\langle (x_1, x_2), (y_1, y_2) \rangle = 3x_1y_1 + x_2y_2$. Justify using axioms. **(4 marks)**

Question 8 (10 Marks)

a) Define the norm of a vector in an inner product space and express it in terms of the inner product. **(2 marks)**
b) For the vectors $u = (1, 2, 3)$ and $v = (4, 5, 6)$ in R^3 with the standard inner product, calculate $||u||$, $||v||$, and $||u + v||$. **(4 marks)**
c) Prove that for any vector v in an inner product space, $||\alpha v|| = |\alpha|\ ||v||$ where α is a scalar. Discuss the geometric significance of this property. **(4 marks)**

Question 9 (10 Marks)

a) Define linear dependence and linear independence of vectors. Provide one example of each. **(3 marks)**

b) Test whether the vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 3, 4)$, and $v_3 = (3, 5, 7)$ are linearly dependent or independent. If dependent, express one vector as a linear combination of others. **(4 marks)**

c) Show that if a set $S = \{v_1, v_2, ..., v_n\}$ contains the zero vector, then S is linearly dependent. Extend this to prove that any set containing a linearly dependent subset is itself linearly dependent. **(3 marks)**

Question 10 (10 Marks)

a) List the steps involved in the Gram-Schmidt orthogonalization process. **(2 marks)**
b) Apply the Gram-Schmidt process to orthogonalize the vectors $v_1 = (1, 1, 0)$, $v_2 = (1, 0, 1)$, and $v_3 = (0, 1, 1)$ in R^3. **(5 marks)**
c) Design a modified Gram-Schmidt algorithm that produces an orthonormal set directly (without a separate normalization step at the end). Explain why this might be computationally advantageous. **(3 marks)**

Question 11 (10 Marks)

a) For the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find: (i) All eigenvalues (ii) Corresponding eigenvectors for each eigenvalue (iii) Verify that $Av = \lambda v$ for each eigenvalue-eigenvector pair **(3 marks)**

b) Determine the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 3 & 2 & 2 \\ 4 & 2 & 5 & 7 \end{pmatrix}$ by: (i) Finding the dimension of row space (ii) Finding the dimension of column space (iii) Verifying both methods yield the same result **(5 marks)**

c) Verify whether the function $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$ defines an inner product on R^2, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. **(2 marks)**