25MT103

LINEAR ALGEBRA

UNIT -1: MATRICES

Dr. D Bhanu Prakash

Assistant Professor

VIGNAN'S
FOUNDATION FOR SCIENCE, TECHNOLOGY & RESEARCH
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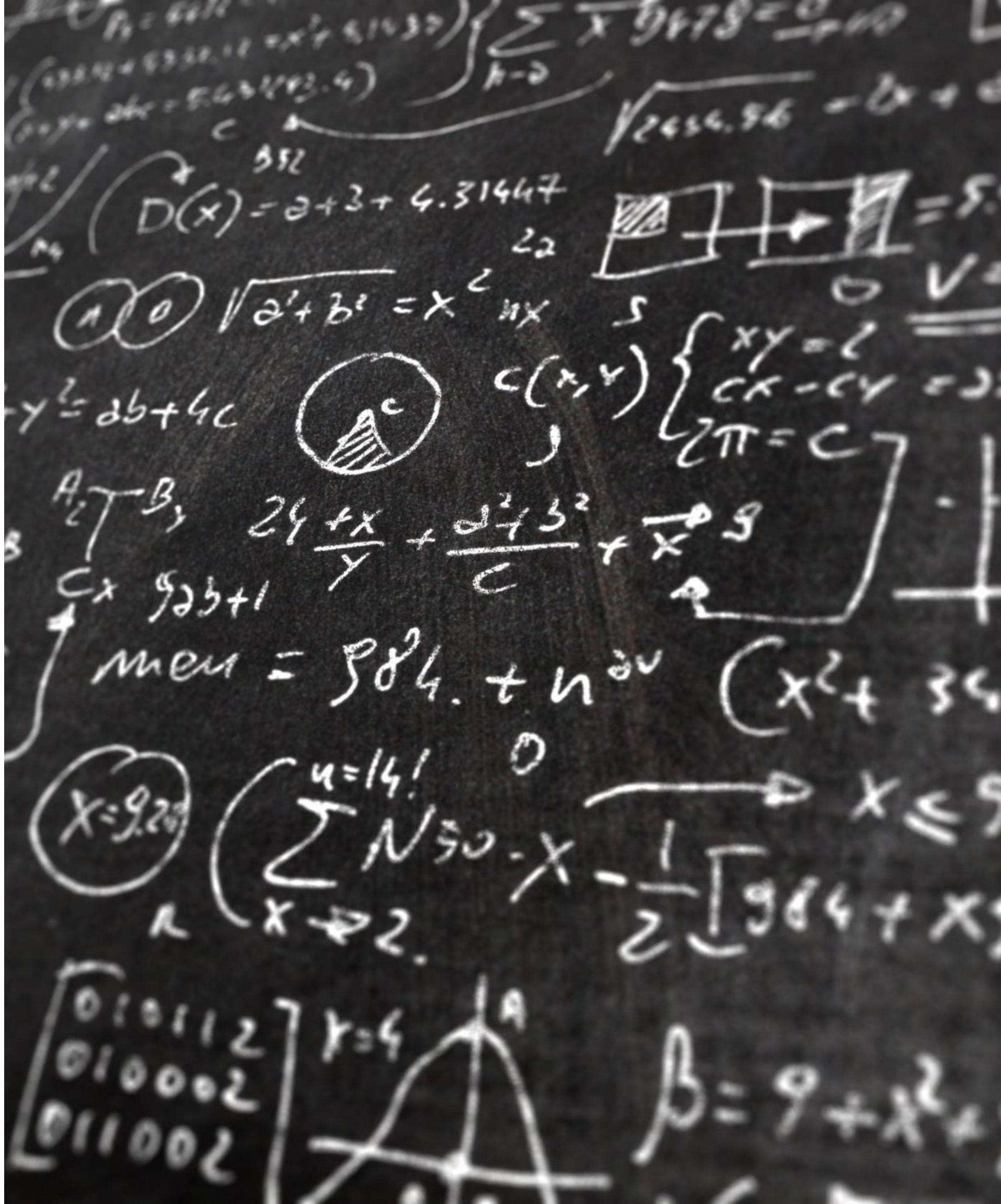

MATRICES:

Elementary row and column operations, Elementary matrices, Similar Matrices, Echelon form, Row reduced echelon form, Rank of a matrix, Inverse of a matrix by Gauss-Jordan method, LU decomposition.

SYLLABUS

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

Row Operations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

So we need to do row operations that make the start matrix look like the end one.

Strategy (preliminary): fiddle with it so we only have ones and zeros. [animated]

Gaussian Elimination Method

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that
would produce ugly fractions.

Let's swap the last two rows first.

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{aligned} x \quad\quad &= 1 \\ y \quad &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2 \cdot (-2) + 3 \cdot 3 &= 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned} \checkmark$$

Row Equivalence

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Summary

- Solving a system of equations means producing all values for the unknowns that make all the equations true simultaneously.
- It is easier to solve a system of linear equations if you put all the coefficients in an **augmented matrix**.
- Solving a system using the elimination method means doing **elementary row operations** on an augmented matrix.
- Two systems or matrices are **row-equivalent** if one can be obtained from the other by doing a sequence of elementary row operations. Row-equivalent systems have the *same solution set*.
- A linear system with no solutions is called **inconsistent**.
- The (reduced) row echelon form of a matrix is its "solved" row-equivalent version.

(Reduced) Row Echelon Form

Review from last time

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Row echelon form:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\star}$ = **pivots**

Poll

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right)$

Answer: B, D, E, F.

Note that A is in row echelon form though.

Inconsistent Matrices

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

An augmented matrix corresponds to an inconsistent system of equations if and only if *the last* (i.e., the augmented) *column is a pivot column.*

Free Variables

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array}\right).$$

In this matrix:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

the free variables are x_2 and x_4. (What about the last column?)

Poll

Poll

Is it possible for a system of linear equations to have exactly two solutions?

Trichotomy

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   In this case, the system has a *unique solution*. Picture:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables, and you get a unique solution.
- A linear system has zero, one, or infinitely many solutions.

Rank of a Matrix (rank(A)) : the number of nonzero rows in a row echelon form of the matrix A.

Q:. Determine the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{REF}(A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem: Let $A\mathbf{x} = \mathbf{b}$ be a system of equations with n variables.

1. if $\text{rank}(A) = \text{rank}([A \;\; \mathbf{b}]) = n$, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.
2. if $\text{rank}(A) = \text{rank}([A \;\; \mathbf{b}]) < n$, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
3. If $\text{rank}(A) \neq \text{rank}([A \;\; \mathbf{b}])$, then the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Theorem. If a homogeneous linear system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n-r$ free variables.

Theorem. A homogeneous linear system with more unknowns than equations has infinitely many solutions.

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What is LU Decomposition?

LU decomposition was introduced by mathematician Tadeusz Banachiewicz in 1938. In numerical analysis and linear algebra, LU decomposition of a matrix is the factorization of a given square matrix into two triangular matrices, one upper triangular matrix and another lower triangular matrix, such that the product of these two matrices gives the original matrix.

Assume→

$[A]$ = Original matrix

$[L]$ = Lower triangular matrix

$[U]$ = Upper triangular matrix

$$[A] = [L][U]$$

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LU Decomposition Method

$$[A]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

1. Decompose

$$[A]=[L][U] \qquad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Cont…

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LU Decomposition Method

2. Forward substitution:

Given **[L]** and **[B]** find **[Y]**

$$[L][Y]=[B] \quad \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

3. Backward substitution

Given **[U]** and **[Y]** find **[X]**

$$[U][X]=[Y] \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Example

Problem:

Solve the following system of equations by LU decomposition.

$$x_1 + 7x_2 - 4x_3 = -51$$
$$4x_1 - 4x_2 + 9x_3 = 62$$
$$12x_1 - x_2 + 3x_3 = 8$$

Solution:

$$[A] = \begin{bmatrix} 1 & 7 & -4 \\ 4 & -4 & 9 \\ 12 & -1 & 3 \end{bmatrix}$$

Row2-Row1 ×4=Row2

Row3-Row1 ×12=Row3

Cont…

Example

$$=\begin{bmatrix}1 & 7 & -4\\0 & -32 & 25\\0 & -85 & 51\end{bmatrix} \quad Row3-Row2\times\frac{85}{32}=Row3$$

$$\therefore [U]=\begin{bmatrix}1 & 7 & -4\\0 & -32 & 25\\0 & 0 & -15.41\end{bmatrix}$$

$$\therefore [L]=\begin{bmatrix}1 & 0 & 0\\4 & 1 & 0\\12 & \frac{85}{32} & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\4 & 1 & 0\\12 & 2.656 & 1\end{bmatrix}$$

Cont…

Now use forward substitution:

$$[L][Y]=[B] \ for \ [Y] \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 12 & 2.656 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -51 \\ 62 \\ 8 \end{bmatrix}$$

$$\therefore y_1 = -51$$

$$\therefore y_2 = 62 + 204$$

$$= 266$$

$$\therefore y_3 = 8 - 706.496 + 612$$

$$= -86.496$$

$$[Y] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -51 \\ 266 \\ -86.496 \end{bmatrix}$$

Cont…

Now use backward substitution:

$$[U][X]=[Y] \ for \ [X] = \begin{bmatrix} 1 & 7 & -4 \\ 0 & -32 & 25 \\ 0 & 0 & -15.41 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -51 \\ 266 \\ -86.496 \end{bmatrix}$$

$$\therefore x_3 = \frac{86.496}{15.41}$$

$$= 5.61$$

$$\therefore x_2 = \frac{266 - 140.25}{-32}$$

$$= -3.92$$

$$\therefore x_1 = -51 + 27.44 + 22.44$$

$$= -1.12$$

$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1.12 \\ -3.92 \\ 5.61 \end{bmatrix}$$

Why LU Decomposition?

Total computational time for LU Decomposition is proportional to→ $\frac{n^3}{3}+n^2$

Total computation time for Gauss Elimination is proportional to→ $\frac{n^3}{3}+\frac{n^2}{2}$

Now, if n=100

LU Decompostion

$$\frac{n^3}{3}+n^2=\frac{100^3}{3}+100^2$$
$$=343333.3333$$

Gauss Elimination

$$\frac{n^3}{3}+\frac{n^2}{2}=\frac{100^3}{3}+\frac{100^2}{2}$$
$$=338333.3333$$

So→ LU Decomposition > Gauss Elimination

Cont…

Why LU Decomposition?

If the [B] vector changes then what will be happed…

Let m = the number of times the [B] vector changes

The computational times are proportional to

LU decomposition = $\frac{n^3}{3} + m(n^2)$

Gauss Elimination = $m(\frac{n^3}{3} + \frac{n^2}{2})$

LU Decomposition = 8.33×10^5

Gauss Elimination = 1.69×10^7

Now→ LU Decomposition < Gauss Elimination

Gauss-Jordan Method for Inverses

Main Procedure…

Step 1: Write down the matrix *A*, and on its right write an identity matrix of the same size.

Step 2: Perform elementary row operations on the left-hand matrix so as to transform it into an identity matrix. These same operations are performed on the right-hand matrix.

Step 3: When the matrix on the left becomes an identity matrix, the matrix on the right is the desired inverse.

Example...

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 8 & -3 & 5 \\ 7 & -2 & 4 \end{bmatrix}.$$

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 8 & -3 & 5 \\ 7 & -2 & 4 \end{bmatrix}.$$

Step 1: First take identity matrix of same size on it's right side.

$$\sim \left[\begin{array}{ccc|ccc} 4 & -2 & 3 & 1 & 0 & 0 \\ 8 & -3 & 5 & 0 & 1 & 0 \\ 7 & -2 & 4 & 0 & 0 & 1 \end{array}\right]$$

Step 2: In this step we want to make first element of first raw 1 and make 0 below this first element.

So take

$R_2 - 2R_1 \sim \left[\begin{array}{ccc|ccc} 4 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 7 & -2 & 4 & 0 & 0 & 1 \end{array}\right]$

$C_1 - C_3 \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 1 & 0 \\ 3 & -2 & 4 & -1 & 0 & 1 \end{array}\right]$

Step 3: Then make second and third element of first row 0 using column operation.

$$R_2 - R_1 \ \& \ R_3 - 3R_1 \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -3 & 1 & 0 \\ 0 & 4 & -5 & -4 & 0 & 1 \end{array}\right]$$

$$C_2 + 2C_1 \ \& \ C_3 - C_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 3 & -4 & -3 & 1 & 0 \\ 0 & 4 & -5 & -4 & 0 & 1 \end{array}\right]$$

Step 4: Make second element of second row 1.

$$R_2 - R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & -1 & 1 & 1 & 1 & -1 \\ 0 & 4 & -5 & -4 & 0 & 1 \end{array}\right]$$

$$-1R_2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & -1 & -1 & -1 & 1 \\ 0 & 4 & -5 & -4 & 0 & 1 \end{array}\right]$$

Step 5: Make 0 above and below of second element of second row.

$$R_3 - 4R_2 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & 4 & -3 \end{array}\right]$$

Step 6: Take column operation.

$$R_2 - R_3 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & -5 & 1 \\ 0 & 0 & -1 & 0 & 4 & 4 \end{array}\right]$$

Step 7: Make third element of third row 1.

$$-1R_3 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & -5 & 1 \\ 0 & 0 & 1 & 0 & -4 & -4 \end{array}\right]$$

So the matrix right hand side of identity matrix is inverse of given matrix.

$$\text{So} \quad A^{-1} = \begin{bmatrix} 1 & 2 & -3 \\ -1 & -5 & 1 \\ 0 & -4 & -4 \end{bmatrix}$$

Similarity

Definition. An $n \times n$ matrix B is said to be **similar** to an $n \times n$ matrix A if $\boxed{B = S^{-1}AS}$ for some nonsingular $n \times n$ matrix S.

Remark. Two $n \times n$ matrices are similar if and only if they represent the same linear operator on $\mathbb{R}^n$ with respect to different bases.

Theorem Similarity is an *equivalence relation*, which means that

(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Theorem Similarity is an *equivalence relation*, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

Proof: **(i)** $A = I^{-1}AI$.
(ii) If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$
$= S_1^{-1}BS_1$, where $S_1 = S^{-1}$.
(iii) If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
$A = S^{-1}(T^{-1}CT)S = (S^{-1}T^{-1})C(TS) = (TS)^{-1}C(TS)$
$= S_2^{-1}CS_2$, where $S_2 = TS$.

Theorem If A and B are similar matrices then they have the same **(i)** determinant, **(ii)** trace = the sum of diagonal entries, **(iii)** rank, and **(iv)** nullity.