25MT103: Linear Algebra

Unit 2: Systems of Linear Equations

Dr. D Bhanu Prakash
Course Page: dbhanuprakash233.github.io/LA

Assistant Professor,
Department of Mathematics and Statistics.
Contact: db_maths@vignan.ac.in.
dbhanuprakash233.github.io.

VIGNAN'S
FOUNDATION FOR SCIENCE, TECHNOLOGY & RESEARCH
(Deemed to be University) - Estd. u/s 3 of UGC Act 1956

Linear Systems - Lecture Slides

Syllabus

- Systems of Linear Equations
- Matrix Representation
- Consistency using rank
- Gaussian Elimination
- Gauss-Jordan method
- Do-little method

Outline

1. Definitions
2. Solution Methods
   - Gaussian Elimination
   - Gauss–Jordan
   - Doolittle (LU)

Outline

1. Definitions
2. Solution Methods
   - Gaussian Elimination
   - Gauss–Jordan
   - Doolittle (LU)

Linear Equation

Definition

A linear equation is an equality where each term is either a constant or a constant multiplied by a variable (no products or powers of variables). It is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Linear Equation

Definition

A linear equation is an equality where each term is either a constant or a constant multiplied by a variable (no products or powers of variables). It is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Examples

1. $2x + 3y = 5$
2. $4x - y = 1$
3. $x = 0$

Linear System

Definition

A *linear system* consists of one or more linear equations. In n variables, it takes the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \quad \ldots, \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.$$

Linear System

Definition

A *linear system* consists of one or more linear equations. In n variables, it takes the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \quad \ldots, \quad a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.$$

Example

$$2x + 3y = 5$$
$$4x - y = 1$$

represents a 2x2 linear system. Solutions correspond to intersection points of lines in 2D space.

Matrix Representation of Linear Systems

Representation

The system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

can be compactly written as $A\mathbf{x} = \mathbf{b}$, with

$$A = (a_{ij})_{m\times n}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Example

For the system $2x+3y=5,\ 4x-y=1,$

$$A=\begin{pmatrix}2 & 3\\ 4 & -1\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}x\\ y\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}5\\ 1\end{pmatrix}.$$

System & Solution Space

System of Linear Equations

A collection of linear equations in variables $x_1, \dots, x_n$; can be written as

$$A\mathbf{x} = \mathbf{b}, \qquad A \in \mathbb{R}^{m\times n},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m.$$

System & Solution Space

System of Linear Equations

A collection of linear equations in variables $x_1, \ldots, x_n$; can be written as

$$A\mathbf{x} = \mathbf{b}, \qquad A \in \mathbb{R}^{m\times n},\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m.$$

Solution Types

- *Unique solution* if $\mathrm{rank}(A) = \mathrm{rank}([A|\mathbf{b}]) = n$.
- *No solution* (inconsistent) if $\mathrm{rank}(A) < \mathrm{rank}([A|\mathbf{b}])$.
- *Infinitely many solutions* if $\mathrm{rank}(A) = \mathrm{rank}([A|\mathbf{b}]) < n$ (free variables exist).

ERO and Rank

ERO and Rank

Elementary Row Operations

Three types: (1) swap two rows, (2) multiply a row by nonzero scalar, (3) add a multiple of one row to another. These preserve solution set.

Rank

rank(M) is the number of leading 1s (pivot columns) in row-echelon form of M, equivalently dimension of column space. Use rank to determine consistency.

Rank and Consistency

Augmented Matrix

Represent the system $A\mathbf{x} = \mathbf{b}$ by the augmented matrix $[A|\mathbf{b}]$ obtained by appending $\mathbf{b}$ as an extra column to A. Row operations on $[A|\mathbf{b}]$ correspond to equivalent systems.

Consistency Criterion

System is consistent iff $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$. If consistent and rank $= n$, unique solution.

Examples

Case 1

Calculate the number of solutions for

$$\begin{aligned} x+2y-z &= 3 \\ 2x-y+3z &= 7 \\ 3x+y+2z &= 10 \end{aligned}$$

Examples

Case 1

Calculate the number of solutions for

$$
\begin{aligned}
x + 2y - z &= 3 \\
2x - y + 3z &= 7 \\
3x + y + 2z &= 10
\end{aligned}
$$

Unique Solution.

Examples

Case 2

$$
\begin{aligned}
x+y+z &= 6 \\
2x+2y+2z &= 12 \\
x-y+0z &= 0
\end{aligned}
$$

Examples

Case 2

$$x + y + z = 6$$
$$2x + 2y + 2z = 12$$
$$x - y + 0z = 0$$

Infinitely many solutions.

Examples

Case 3

$$x + y + z = 3$$
$$2x + 2y + 2z = 6$$
$$x + y + z = 4$$

Examples

Case 3

$$x+y+z=3$$
$$2x+2y+2z=6$$
$$x+y+z=4$$

No Solution.

Examples

Case 4

$$x + 2y + 3z = 1$$
$$2x + 4y + 6z = 2$$

Examples

Case 4

$$
\begin{aligned}
x + 2y + 3z &= 1 \\
2x + 4y + 6z &= 2
\end{aligned}
$$

Infinitely many solutions.

Outline

1. Definitions
2. Solution Methods
   - Gaussian Elimination
   - Gauss–Jordan
   - Doolittle (LU)

Worked Example: Problem Statement

Solve the system:

$$\begin{aligned} x+y+z &= 6 \\ 2x+3y+z &= 14 \\ x-y+2z &= 2 \end{aligned}$$

We will solve this system by (1) Gaussian elimination, (2) Gauss–Jordan, and (3) Doolittle (LU) decomposition — step by step.

Gaussian Elimination

Gaussian Elimination

Reduce $[A|\mathbf{b}]$ to *row-echelon form* (upper triangular) using elementary row operations; then use back-substitution to find unknowns.

Gaussian Elimination

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 14 \\ 1 & -1 & 2 & 2 \end{array}\right]$$

Gaussian Elimination

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 14 \\ 1 & -1 & 2 & 2 \end{array}\right]$$

Use $R_2 \leftarrow R_2 - 2R_1$ and $R_3 \leftarrow R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & -4 \end{array}\right]$$

Gaussian Elimination

Pivot in row2 is 1. Use $R_3 \leftarrow R_3 + 2R_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Gaussian Elimination

Pivot in row2 is 1. Use $R_3 \leftarrow R_3 + 2R_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Back-substitution:
from row3, $-z = 0 \Rightarrow z = 0$.
Row2: $y - z = 2 \Rightarrow y = 2$.
Row1: $x + y + z = 6 \Rightarrow x = 4$.
Unique solution: $(x, y, z) = (4, 2, 0)$.

Gauss–Jordan Method

Gauss–Jordan

Reduce $[A|\mathbf{b}]$ to reduced row-echelon form (RREF) so each pivot is 1 and is the only nonzero entry in its column. Solutions are read directly; no back-substitution required.

Gauss–Jordan: Start with augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 14 \\ 1 & -1 & 2 & 2 \end{array}\right]$$

Gauss–Jordan: Make zeros below and above pivots

First eliminate below pivot in column1: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$ giving

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & -4 \end{array}\right]$$

Gauss–Jordan: Make zeros below and above pivots

First eliminate below pivot in column1: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$ giving

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & -4 \end{array}\right]$$

Then make pivot in row2 the only nonzero in its column: add $2\times$ row2 to row3 and subtract row2 from row1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Gauss–Jordan: Make zeros below and above pivots

First eliminate below pivot in column1: $R_2 \leftarrow R_2 - 2R_1$, $R_3 \leftarrow R_3 - R_1$ giving

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 1 & -4 \end{array}\right]$$

Then make pivot in row2 the only nonzero in its column: add $2\times$ row2 to row3 and subtract row2 from row1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Finally scale row3 by -1 to make pivot 1 and eliminate above: $R_3 \leftarrow -R_3$ then $R_1 \leftarrow R_1 - 2R_3$, $R_2 \leftarrow R_2 + R_3$ leads to RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Doolittle (LU) Method

Doolittle's LU Decomposition

If A is square and can be decomposed as $A = LU$ where L is unit lower-triangular (1s on diagonal) and U is upper-triangular, solve by

$$L\mathbf{y} = \mathbf{b} \quad \text{(forward substitution)}, \qquad U\mathbf{x} = \mathbf{y} \quad \text{(back substitution)}.$$

Doolittle constructs L with unit diagonal and computes entries row-by-row.

Doolittle: Setup

We want $A = LU$ with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}.$$

Doolittle computes rows of U and columns of L sequentially.

Doolittle: Compute LU

From $u_{1j} = a_{1j}$: $u_{11} = 1, u_{12} = 1, u_{13} = 1$. Compute $\ell_{21} = a_{21}/u_{11} = 2/1 = 2$, $\ell_{31} = a_{31}/u_{11} = 1/1 = 1$.

Doolittle: Compute LU

From $u_{1j} = a_{1j}$: $u_{11} = 1, u_{12} = 1, u_{13} = 1$. Compute $\ell_{21} = a_{21}/u_{11} = 2/1 = 2$, $\ell_{31} = a_{31}/u_{11} = 1/1 = 1$.
Compute $u_{22} = a_{22} - \ell_{21}u_{12} = 3 - 2 \cdot 1 = 1$.
Compute $u_{23} = a_{23} - \ell_{21}u_{13} = 1 - 2 \cdot 1 = -1$.
Compute $\ell_{32} = (a_{32} - \ell_{31}u_{12})/u_{22} = (-1 - 1 \cdot 1)/1 = -2$.

Doolittle: Compute LU

From $u_{1j} = a_{1j}$: $u_{11} = 1, u_{12} = 1, u_{13} = 1$. Compute $\ell_{21} = a_{21}/u_{11} = 2/1 = 2$, $\ell_{31} = a_{31}/u_{11} = 1/1 = 1$.
Compute $u_{22} = a_{22} - \ell_{21}u_{12} = 3 - 2 \cdot 1 = 1$.
Compute $u_{23} = a_{23} - \ell_{21}u_{13} = 1 - 2 \cdot 1 = -1$.
Compute $\ell_{32} = (a_{32} - \ell_{31}u_{12})/u_{22} = (-1 - 1 \cdot 1)/1 = -2$.
Compute

$$u_{33} = a_{33} - \ell_{31}u_{13} - \ell_{32}u_{23} = 2 - 1 \cdot 1 - (-2) \cdot (-1) = 2 - 1 - 2 = -1.$$

So

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Doolittle: Solve $L\mathbf{y} = \mathbf{b}$ (forward sub)

Solve

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ 2 \end{pmatrix}.$$

Forward substitution: $y_1 = 6$. $2y_1 + y_2 = 14 \Rightarrow y_2 = 14 - 2 \cdot 6 = 2$.
$y_1 - 2y_2 + y_3 = 2 \Rightarrow y_3 = 2 - 6 + 4 = 0$. So $\mathbf{y} = (6, 2, 0)^T$.

Doolittle: Solve $U\mathbf{x} = \mathbf{y}$ (back sub)

Solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}.$$

Back substitution: from row3 $-z = 0 \Rightarrow z = 0$. Row2: $y - z = 2 \Rightarrow y = 2$. Row1: $x + y + z = 6 \Rightarrow x = 4$.

Thank You!

Dr. D Bhanu Prakash
dbhanuprakash233.github.io
Mail: db_maths@vignan.ac.in