SCHOOL OF APPLIED SCIENCE & HUMANITIES

DEPARTMENT OF MATHEMATICS

Subject: Linear Algebra | Subject Code : 25MT103

Sem. : I | Academic Year: 2025-2026

Section: 7, 14, 21 | Regulation: R25

Problem Set – Unit 3

Question 1 (10 Marks)

a) Define the characteristic equation of a matrix. Explain how eigenvalues are obtained from the characteristic equation. **(2 marks)**

b) Find the characteristic equation and eigenvalues of the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$. Verify your eigenvalues by substituting back into the characteristic equation. **(4 marks)**

c) Show that the sum of eigenvalues of a matrix equals its trace, and the product of eigenvalues equals its determinant. Demonstrate this property with a 3×3 matrix of your choice. **(4 marks)**

Question 2 (10 Marks)

a) Define an eigenvector and eigenvalue of a matrix. Explain the relationship between them using the equation $Av = \lambda v$. **(2 marks)**

b) For the matrix $A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$, find: (i) All eigenvalues (ii) Corresponding eigenvectors for each eigenvalue (iii) Verify that $Av = \lambda v$ for each eigenvalue-eigenvector pair **(5 marks)**

c) Analyze why the zero vector is never considered an eigenvector, even though $A(0) = \lambda(0)$ for any λ. Create an example showing how different eigenvectors can correspond to the same eigenvalue. **(3 marks)**

Question 3 (10 Marks)

a) State any four important properties of eigenvalues and eigenvectors. **(3 marks)**

b) Given the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, determine: (i) The eigenvalues of A (ii) The eigenvalues of A^2, A^{-1}, and 3A (iii) Verify the relationship between eigenvalues of A and its powers **(4 marks)**

c) Consider the above matrix A. Prove that if λ is an eigenvalue of an invertible matrix A with eigenvector v, then $1/\lambda$ is an eigenvalue of A^{-1} with the same eigenvector v. Discuss the implications when A is singular. **(3 marks)**

Question 4 (10 Marks)

a) Explain the concept of algebraic multiplicity and geometric multiplicity of an eigenvalue. **(2 marks)**

b) For the matrix $A = \begin{pmatrix} 5 & -2 & -4 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{pmatrix}$, find: (i) All eigenvalues and their algebraic multiplicities (ii) The eigenvectors corresponding to each eigenvalue (iii) The geometric multiplicity of each eigenvalue **(5 marks)**

c) Analyze the relationship between algebraic and geometric multiplicities. Construct an example where they are equal and another where they differ. **(3 marks)**

Question 5 (10 Marks)

a) State the Cayley-Hamilton Theorem. Explain what it means for a matrix to satisfy its own characteristic equation. **(2 marks)**

b) Verify the Cayley-Hamilton Theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Use this result to compute A^3. **(4 marks)**

c) Using the Cayley-Hamilton Theorem, derive a general method to express higher powers of a matrix (A^n for large n) in terms of lower powers. Demonstrate with a specific example for $n = 5$. **(4 marks)**

Question 6 (10 Marks)

a) Describe the process of finding the inverse of a matrix using the Cayley-Hamilton Theorem. **(2 marks)**

b) Apply the Cayley-Hamilton Theorem to find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Verify your answer by computing AA^{-1}. **(5 marks)**

c) Compare and contrast three methods of finding matrix inverse: (i) Adjoint method, (ii) Row reduction, and (iii) Cayley-Hamilton Theorem. Evaluate which method is most efficient for different types of matrices. **(3 marks)**

Question 7 (10 Marks)

a) Define a diagonalizable matrix. State the necessary and sufficient condition for a matrix to be diagonalizable. **(2 marks)**

b) Determine whether the matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ is diagonalizable. If yes, find matrices P and D such that $A = PDP^{-1}$. If no, justify your answer. **(4 marks)**

c) Prove that if an n×n matrix has n distinct eigenvalues, then it is diagonalizable. Construct a counterexample showing that a matrix can be diagonalizable even without n distinct eigenvalues. **(4 marks)**

Question 8 (10 Marks)

a) Explain the relationship between eigenvectors and the matrix P used in diagonalization $A = PDP^{-1}$. What does matrix D represent? **(2 marks)**

b) Diagonalize the matrix $A = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}$. Find matrices P and D such that $A = PDP^{-1}$, and verify your result by computing PDP^{-1}. **(5 marks)**

c) Design a step-by-step algorithm to test if a given n×n matrix is diagonalizable and, if so, to find the diagonalization. Apply your algorithm to $A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. **(3 marks)**

Question 9 (10 Marks)

a) State the theorem relating linear independence of eigenvectors corresponding to distinct eigenvalues. **(2 marks)**

b) Given the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$, diagonalize A and use the diagonalization to compute A^5. **(5 marks)**

c) Prove that eigenvectors corresponding to distinct eigenvalues are linearly independent. Discuss why this property is crucial for diagonalization. **(3 marks)**

Question 10 (10 Marks)

a) Explain how diagonalization simplifies the computation of powers of a matrix. Write the general formula for A^n in terms of P, D, and P^{-1}. **(2 marks)**

b) For the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$, find: (i) The diagonalization $A = PDP^{-1}$ (ii) A^{10} using the diagonalization (iii) Verify your result by computing A^2 directly and checking consistency **(5 marks)**

c) Create a practical application problem involving population dynamics or Fibonacci sequences where computing high powers of a matrix is required. Solve it using diagonalization and evaluate the efficiency compared to direct multiplication. **(3 marks)**

Question 11 (10 Marks)

a) Recall the properties of symmetric matrices with respect to eigenvalues and eigenvectors. **(2 marks)**

b) Consider the matrix A = $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Show that: (i) All eigenvalues are real (ii) Eigenvectors corresponding to distinct eigenvalues are orthogonal (iii) The matrix is diagonalizable **(5 marks)**

c) Prove that if A is a symmetric matrix, then eigenvectors corresponding to distinct eigenvalues are orthogonal. Analyze how this property leads to orthogonal diagonalization. **(3 marks)**

Question 12 (10 Marks)

a) Define similar matrices and explain how similarity relates to eigenvalues. **(2 marks)**

b) Given matrices A = $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$: (i) Find eigenvalues of both A and B (ii) Determine if A and B are similar (iii) If similar, find the matrix P such that $B = P^{-1}AP$ (iv) Use diagonalization to compute A^8 **(5 marks)**

c) Evaluate the statement: "Two matrices are similar if and only if they have the same eigenvalues." Is this statement true? Provide rigorous justification with examples and counterexamples. Design a complete criterion for determining when two matrices are similar. **(3 marks)**