25MT103: Linear Algebra

Unit 3: Eigenvalues, Eigenvectors and Diagonalization

Dr. D Bhanu Prakash
Course Page: dbhanuprakash233.github.io/LA

Assistant Professor,
Department of Mathematics and Statistics.
Contact: db_maths@vignan.ac.in.
dbhanuprakash233.github.io.

VIGNAN'S FOUNDATION FOR SCIENCE, TECHNOLOGY & RESEARCH
(Deemed to be University) - Estd. u/s 3 of UGC Act 1956

Eigenvalues, Eigenvectors - Lecture Slides

Syllabus

- ☞ Characteristic Equation
- ☞ Eigenvalues, Eigenvectors and their Properties
- ☞ Cayley-Hamilton Theorem
- ☞ Diagonalization of a Matrix (only for diagonalizable matrices)
- ☞ Inverse of a matrix by Cayley-Hamilton Theorem
- ☞ Power of a diagonalizable square matrix

Outline

Outline

Linear Independence

Definition: Linear Independence

A set $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is **linearly independent** if the only scalars $c_1, \ldots, c_k$ with

$$c_1 v_1 + \cdots + c_k v_k = 0$$

are $c_1 = \cdots = c_k = 0$. Otherwise the set is **linearly dependent**.

Linear Independence

Definition: Linear Independence

A set $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is **linearly independent** if the only scalars $c_1, \ldots, c_k$ with

$$c_1 v_1 + \cdots + c_k v_k = 0$$

are $c_1 = \cdots = c_k = 0$. Otherwise the set is **linearly dependent**.

Quick Example (2D)

The vectors $\{(1,1)^\top, (1,-1)^\top\}$ are linearly independent because the 2×2 matrix with these columns has nonzero determinant.

Linear Independence

Definition: Linear Independence

A set $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ is **linearly independent** if the only scalars $c_1, \ldots, c_k$ with

$$c_1 v_1 + \cdots + c_k v_k = 0$$

are $c_1 = \cdots = c_k = 0$. Otherwise the set is **linearly dependent**.

Quick Example (2D)

The vectors $\{(1,1)^\top, (1,-1)^\top\}$ are linearly independent because the 2×2 matrix with these columns has nonzero determinant.

Quick Example (3D)

The vectors $\{(1,0,0)^\top, (0,1,0)^\top, (1,1,0)^\top\}$ are linearly dependent (third is sum of first two).

Characteristic Equation and Eigenvalues

Definition: Characteristic Equation

For square $A \in \mathbb{R}^{n\times n}$, the characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I).$$

The scalar roots of $p_A(\lambda) = 0$ are the eigenvalues of A.

Characteristic Equation and Eigenvalues

Definition: Characteristic Equation

For square $A \in \mathbb{R}^{n\times n}$, the characteristic polynomial is

$$p_A(\lambda) = \det(A - \lambda I).$$

The scalar roots of $p_A(\lambda) = 0$ are the eigenvalues of A.

2×2 Example

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$. Roots: $\lambda = 3, 1$.

Eigenvectors and Properties

Definition: Eigenvector

An eigenvector $x \neq 0$ associated to eigenvalue λ satisfies

$$(A - \lambda I)x = 0.$$

The set of all such x (with zero) is the eigenspace $\mathrm{Null}(A - \lambda I)$.

Eigenvectors and Properties

Definition: Eigenvector

An eigenvector $x \neq 0$ associated to eigenvalue λ satisfies

$$(A - \lambda I)x = 0.$$

The set of all such x (with zero) is the eigenspace $\mathrm{Null}(A - \lambda I)$.

Useful Properties

- Eigenvectors for distinct eigenvalues are linearly independent.
- For triangular matrices, eigenvalues are diagonal entries.
- Sum of eigenvalues = $\mathrm{tr}(A)$; product = $\det(A)$.

Algebraic Multiplicity and Geometric Multiplicity

Definitions

- **Algebraic multiplicity (AM)** of eigenvalue λ: multiplicity of λ as a root of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$.
- **Geometric multiplicity (GM)** of λ: dimension of the eigenspace $\mathrm{Null}(A - \lambda I)$.

Algebraic Multiplicity and Geometric Multiplicity

Definitions

- **Algebraic multiplicity (AM)** of eigenvalue λ: multiplicity of λ as a root of the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$.
- **Geometric multiplicity (GM)** of λ: dimension of the eigenspace $\mathrm{Null}(A - \lambda I)$.

Important facts

For each eigenvalue λ,

$$1 \leq \mathrm{GM}(\lambda) \leq \mathrm{AM}(\lambda).$$

Matrix A is diagonalizable iff the total number of linearly independent eigenvectors equals n (equivalently $\mathrm{GM}(\lambda) = \mathrm{AM}(\lambda)$ for every eigenvalue).

Cayley–Hamilton Theorem

Cayley–Hamilton

If $p_A(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$, then

$$p_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_0 I = 0.$$

Cayley–Hamilton Theorem

Cayley–Hamilton

If $p_A(\lambda) = \det(A - \lambda I) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0$, then

$$p_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_0 I = 0.$$

2×2 quick use

For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $p_A(\lambda) = \lambda^2 - 4\lambda + 3$. Thus $A^2 - 4A + 3I = 0$; rearrange to get expressions useful for A^{-1} or powers.

Diagonalization and Powers

Diagonalization

A is diagonalizable if there exists invertible P with $P^{-1}AP = D$ (diagonal).
Then $A^k = PD^kP^{-1}$ which is cheap to compute.

Diagonalization and Powers

Diagonalization

A is diagonalizable if there exists invertible P with $P^{-1}AP = D$ (diagonal). Then $A^k = PD^kP^{-1}$ which is cheap to compute.

2×2 example

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenpairs leading to $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, D = \mathrm{diag}(3,1)$, so $A^n = P\mathrm{diag}(3^n, 1^n)P^{-1}$.

Diagonalization and Powers

Diagonalization

A is diagonalizable if there exists invertible P with $P^{-1}AP = D$ (diagonal). Then $A^k = PD^kP^{-1}$ which is cheap to compute.

2×2 example

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenpairs leading to $P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $D = \text{diag}(3, 1)$, so $A^n = P\text{diag}(3^n, 1^n)P^{-1}$.

3×3 remark

For a 3×3 diagonalizable matrix, the same idea applies: diagonalize and raise diagonal entries to the kth power.

Outline

Problem: matrix under study

We study the matrix

$$A = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}.$$

We will compute

- $p_A(\lambda)$
- eigenvalues with algebraic multiplicities
- eigenspaces (and their dimensions)
- eigenvectors (integer-scaled and normalized)
- an explicit inverse P^{-1} of the modal matrix P
- verify diagonalization

Step 1: compute $p_A(\lambda) = \det(A - \lambda I)$

Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -10 & -5 \\ 2 & 14-\lambda & 2 \\ -4 & -8 & 6-\lambda \end{pmatrix}.$$

Step 1: compute $p_A(\lambda) = \det(A - \lambda I)$

Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -10 & -5 \\ 2 & 14-\lambda & 2 \\ -4 & -8 & 6-\lambda \end{pmatrix}.$$

We compute the determinant by cofactor expansion along the first row.

$$\det(A-\lambda I) = (5-\lambda)\det\begin{pmatrix} 14-\lambda & 2 \\ -8 & 6-\lambda \end{pmatrix} - (-10)\det\begin{pmatrix} 2 & 2 \\ -4 & 6-\lambda \end{pmatrix}$$

$$+(-5)\det\begin{pmatrix} 2 & 14-\lambda \\ -4 & -8 \end{pmatrix}.$$

Step 1 (cont.)

$$\det(A-\lambda I) = (5-\lambda)\big((14-\lambda)(6-\lambda)+16\big)+10\big(2(6-\lambda)+8\big)$$
$$-5\big(-16+4(14-\lambda)\big).$$

Step 1 (cont.)

$$\det(A-\lambda I) = (5-\lambda)\big((14-\lambda)(6-\lambda)+16\big)+10\big(2(6-\lambda)+8\big)$$
$$-5\big(-16+4(14-\lambda)\big).$$

Now expand terms carefully (expand each parenthesis and collect like powers of λ). After simplification one obtains

$$\det(A-\lambda I) = (\lambda-10)^2(\lambda-5).$$

Step 1 (cont.)

$$\det(A-\lambda I) = (5-\lambda)\big((14-\lambda)(6-\lambda)+16\big)+10\big(2(6-\lambda)+8\big) \\ -5\big(-16+4(14-\lambda)\big).$$

Now expand terms carefully (expand each parenthesis and collect like powers of λ). After simplification one obtains

$$\det(A-\lambda I) = (\lambda-10)^2(\lambda-5).$$

Hence the characteristic polynomial (up to sign) factors as

$$p_A(\lambda) = (\lambda-10)^2(\lambda-5).$$

Step 1 (conclusion): eigenvalues and algebraic multiplicities

From $p_A(\lambda)$ we read:

$$\lambda_1 = 5, \quad \mathrm{AM}(5) = 1; \qquad \lambda_2 = 10, \quad \mathrm{AM}(10) = 2.$$

Next we compute eigenspaces to find geometric multiplicities and eigenvectors.

Step 2: Eigenspace for $\lambda = 5$ — form $A - 5I$

Compute

$$A - 5I = \begin{pmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Step 2: Eigenspace for $\lambda = 5$ — form $A - 5I$

Compute

$$A - 5I = \begin{pmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

We solve $(A - 5I)x = 0$. Form the augmented system and do row reduction. Below we show row operations to reach RREF.

Step 2 (cont.): Row-reduction to RREF for $\lambda = 5$

Start with the coefficient matrix $(A - 5I)$

$$\begin{pmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Step 2 (cont.): Row-reduction to RREF for $\lambda = 5$

Start with the coefficient matrix $(A-5I)$

$$\begin{pmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Perform elementary row operations (one possible path):

$$R_1 \leftrightarrow R_2 : \begin{pmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ -4 & -8 & 1 \end{pmatrix}$$

Step 2 (cont.): Row-reduction to RREF for $\lambda = 5$

Start with the coefficient matrix $(A - 5I)$

$$\begin{pmatrix} 0 & -10 & -5 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Perform elementary row operations (one possible path):

$$R_1 \leftrightarrow R_2 : \begin{pmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ -4 & -8 & 1 \end{pmatrix}$$

$$R_3 \leftarrow R_3 + 2R_1 : \begin{pmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ 0 & 10 & 5 \end{pmatrix}$$

Step 2 (cont.): Row-reduction to RREF for $\lambda = 5$

$$R_3 \leftarrow R_3 + R_2 : \begin{pmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2 (cont.): Row-reduction to RREF for $\lambda = 5$

$$R_3 \leftarrow R_3 + R_2 : \begin{pmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Scale } R_1 \leftarrow \tfrac{1}{2}R_1,\ R_2 \leftarrow -\tfrac{1}{10}R_2 : \begin{pmatrix} 1 & \frac{9}{2} & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Step 2 (cont.): Row-reduction to RREF for $\lambda = 5$

$$R_3 \leftarrow R_3 + R_2 : \begin{pmatrix} 2 & 9 & 2 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Scale } R_1 \leftarrow \tfrac{1}{2}R_1,\ R_2 \leftarrow -\tfrac{1}{10}R_2 : \begin{pmatrix} 1 & \frac{9}{2} & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_1 \leftarrow R_1 - \tfrac{9}{2}R_2 \ \Rightarrow \ \begin{pmatrix} 1 & 0 & -\frac{5}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

This is the RREF.

Step 2 (cont.): Solve for eigenvectors for $\lambda = 5$

From the RREF equations:

$$x_1 - \frac{5}{4}x_3 = 0, \qquad x_2 + \frac{1}{2}x_3 = 0.$$

Step 2 (cont.): Solve for eigenvectors for $\lambda = 5$

From the RREF equations:

$$x_1 - \tfrac{5}{4}x_3 = 0, \qquad x_2 + \tfrac{1}{2}x_3 = 0.$$

Let $x_3 = t$ (free). Then

$$x = t\begin{pmatrix} 5/4 \\ -1/2 \\ 1 \end{pmatrix} = t'\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix},$$

where in the last expression we scaled by $t' = t/(1/4)$ to get integer entries.

Step 2 (cont.): Solve for eigenvectors for $\lambda = 5$

From the RREF equations:

$$x_1 - \frac{5}{4}x_3 = 0, \qquad x_2 + \frac{1}{2}x_3 = 0.$$

Let $x_3 = t$ (free). Then

$$x = t\begin{pmatrix} 5/4 \\ -1/2 \\ 1 \end{pmatrix} = t'\begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix},$$

where in the last expression we scaled by $t' = t/(1/4)$ to get integer entries. Thus an integer-scaled eigenvector for $\lambda = 5$ is

$$v_5 = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix},$$

and the eigenspace has dimension $\mathrm{GM}(5) = 1$.

Step 3: Eigenspace for $\lambda = 10$ — form $A - 10I$

Compute

$$A - 10I = \begin{pmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{pmatrix}.$$

Step 3: Eigenspace for $\lambda = 10$ — form $A - 10I$

Compute

$$A - 10I = \begin{pmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{pmatrix}.$$

Solve $(A - 10I)x = 0$ by row reduction.

Step 3: Eigenspace for $\lambda = 10$ — form $A - 10I$

Compute

$$A - 10I = \begin{pmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{pmatrix}.$$

Solve $(A - 10I)x = 0$ by row reduction. Observe rows: $R_2 = -\frac{2}{5}R_1$ and $R_3 = \frac{4}{5}R_1$ (so only one independent equation).

Step 3: Eigenspace for $\lambda = 10$ — form $A - 10I$

Compute

$$A - 10I = \begin{pmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{pmatrix}.$$

Solve $(A - 10I)x = 0$ by row reduction. Observe rows: $R_2 = -\frac{2}{5}R_1$ and $R_3 = \frac{4}{5}R_1$ (so only one independent equation).
Bring to RREF (a compact path):

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Step 3: Eigenspace for $\lambda = 10$ — form $A - 10I$

Compute

$$A - 10I = \begin{pmatrix} -5 & -10 & -5 \\ 2 & 4 & 2 \\ -4 & -8 & -4 \end{pmatrix}.$$

Solve $(A - 10I)x = 0$ by row reduction. Observe rows: $R_2 = -\frac{2}{5}R_1$ and $R_3 = \frac{4}{5}R_1$ (so only one independent equation).
Bring to RREF (a compact path):

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence there is a single independent linear equation:

$$x_1 + 2x_2 + x_3 = 0.$$

Step 3 (cont.): Solve for eigenvectors for $\lambda = 10$

From $x_1 = -2x_2 - x_3$, take free parameters $s = x_2,\ t = x_3$. Then

$$x = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Step 3 (cont.): Solve for eigenvectors for $\lambda = 10$

From $x_1 = -2x_2 - x_3$, take free parameters $s = x_2,\ t = x_3$. Then

$$x = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus two independent integer eigenvectors for $\lambda = 10$ are

$$v_{10}^{(1)} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \qquad v_{10}^{(2)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Step 3 (cont.): Solve for eigenvectors for $\lambda = 10$

From $x_1 = -2x_2 - x_3$, take free parameters $s = x_2,\ t = x_3$. Then

$$x = s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Thus two independent integer eigenvectors for $\lambda = 10$ are

$$v_{10}^{(1)} = \begin{pmatrix}-2\\1\\0\end{pmatrix}, \qquad v_{10}^{(2)} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

So $\mathrm{GM}(10) = 2$, matching $\mathrm{AM}(10) = 2$.

Step 4: Collect eigenvectors and check diagonalizability

We have:

$$\lambda = 5: \quad v_5 = (5, -2, 4)^\top, \quad \mathrm{GM}(5) = 1$$

$$\lambda = 10: \quad v_{10}^{(1)} = (-2, 1, 0)^\top,\ v_{10}^{(2)} = (-1, 0, 1)^\top, \quad \mathrm{GM}(10) = 2.$$

Step 4: Collect eigenvectors and check diagonalizability

We have:

$$\lambda = 5: \quad v_5 = (5, -2, 4)^\top, \quad \mathrm{GM}(5) = 1$$

$$\lambda = 10: \quad v_{10}^{(1)} = (-2, 1, 0)^\top,\ v_{10}^{(2)} = (-1, 0, 1)^\top, \quad \mathrm{GM}(10) = 2.$$

Total independent eigenvectors $= 1 + 2 = 3 = n$.
Therefore A is **diagonalizable**.

Step 5: Normalizing the eigenvectors

We now normalize each integer-scaled eigenvector (compute their Euclidean norms and divide).

Step 5: Normalizing the eigenvectors

We now normalize each integer-scaled eigenvector (compute their Euclidean norms and divide).
For $v_5 = (5, -2, 4)^\top$:

$$\|v_5\| = \sqrt{5^2 + (-2)^2 + 4^2} = \sqrt{25 + 4 + 16} = \sqrt{45}.$$

So the unit eigenvector is

$$\hat{v}_5 = \frac{1}{\sqrt{45}} \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}.$$

Step 5: Normalizing the eigenvectors

For $v_{10}^{(1)} = (-2, 1, 0)^\top$:

$$\|v_{10}^{(1)}\| = \sqrt{(-2)^2 + 1^2 + 0^2} = \sqrt{5}, \qquad \hat{v}_{10}^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Step 5: Normalizing the eigenvectors

For $v_{10}^{(1)} = (-2, 1, 0)^\top$:

$$\|v_{10}^{(1)}\| = \sqrt{(-2)^2 + 1^2 + 0^2} = \sqrt{5}, \qquad \hat{v}_{10}^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

For $v_{10}^{(2)} = (-1, 0, 1)^\top$:

$$\|v_{10}^{(2)}\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}, \qquad \hat{v}_{10}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Step 6: Build modal matrix P and diagonal D

If A is diagonalisable, what are P and D is $A = P^{-1}DP$?

Step 6: Build modal matrix P and diagonal D

If A is diagonalisable, what are P and D is $A = P^{-1}DP$?
Take eigenvectors as columns (integer-scaled for P):

$$P = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

Step 6: Build modal matrix P and diagonal D

If A is diagonalisable, what are P and D is $A = P^{-1}DP$?
Take eigenvectors as columns (integer-scaled for P):

$$P = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

We will compute P^{-1} explicitly (so the diagonalization $A = PDP^{-1}$ is fully verified).

Step 7: Compute $\det(P)$ and adjoint for P^{-1}

First compute $\det(P)$. Expand or use standard formula; one gets

$$\det(P) = 5.$$

Step 7: Compute $\det(P)$ and adjoint for P^{-1}

First compute $\det(P)$. Expand or use standard formula; one gets

$$\det(P) = 5.$$

Using the adjugate formula:

$$P^{-1} = \frac{1}{\det(P)}\,\mathrm{adj}(P).$$

We compute the matrix of cofactors (cofactor $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor).

Step 7 (cont.): Cofactors and adjoint (explicit)

Compute minors/cofactors (listed compactly):

$$
\begin{aligned}
M_{11} &= \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1, & C_{11} &= +1, \\
M_{12} &= \det\begin{pmatrix} -2 & 0 \\ 4 & 1 \end{pmatrix} = -2, & C_{12} &= -(-2) = +2, \\
M_{13} &= \det\begin{pmatrix} -2 & 1 \\ 4 & 0 \end{pmatrix} = -4 = -4, & C_{13} &= +(-4) = -4, \\
M_{21} &= \det\begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} = -2, & C_{21} &= -(-2) = +2,
\end{aligned}
$$

Step 7 (cont.): Cofactors and adjoint (explicit)

$$M_{22} = \det\begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} = 5(1) - (-1)(4) = 9, \qquad C_{22} = +9,$$

$$M_{23} = \det\begin{pmatrix} 5 & -2 \\ 4 & 0 \end{pmatrix} = 5(0) - (-2)(4) = 8, \qquad C_{23} = -8,$$

$$M_{31} = \det\begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} = (-2)(0) - (-1)(1) = 1, \quad C_{31} = +1,$$

$$M_{32} = \det\begin{pmatrix} 5 & -1 \\ -2 & 0 \end{pmatrix} = 5(0) - (-1)(-2) = -2, \quad C_{32} = -(-2) = +2,$$

$$M_{33} = \det\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = 5(1) - (-2)(-2) = 1, \qquad C_{33} = +1.$$

Step 7 (cont.): adjoint and P^{-1}

The matrix of cofactors is

$$C = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 9 & -8 \\ 1 & 2 & 1 \end{pmatrix}.$$

Step 7 (cont.): adjoint and P^{-1}

The matrix of cofactors is

$$C = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 9 & -8 \\ 1 & 2 & 1 \end{pmatrix}.$$

The adjoint $\mathrm{adj}(P) = C^\top$ is

$$\mathrm{adj}(P) = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 9 & -8 \\ 1 & 2 & 1 \end{pmatrix}^\top = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Step 7 (cont.): adjoint and P^{-1}

The matrix of cofactors is

$$C = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 9 & -8 \\ 1 & 2 & 1 \end{pmatrix}.$$

The adjoint $\operatorname{adj}(P) = C^\top$ is

$$\operatorname{adj}(P) = \begin{pmatrix} 1 & 2 & -4 \\ 2 & 9 & -8 \\ 1 & 2 & 1 \end{pmatrix}^\top = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Since $\det(P) = 5$,

$$P^{-1} = \frac{1}{5} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 9 & 2 \\ -4 & -8 & 1 \end{pmatrix}.$$

Step 8: Verify diagonalization $P^{-1}AP = D$

Compute $P^{-1}AP$. Using the matrices given:

$$P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{9}{5} & \frac{2}{5} \\ -\frac{4}{5} & -\frac{8}{5} & \frac{1}{5} \end{pmatrix}, \qquad P = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}.$$

Step 8: Verify diagonalization $P^{-1}AP = D$

Compute $P^{-1}AP$. Using the matrices given:

$$P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{9}{5} & \frac{2}{5} \\ -\frac{4}{5} & -\frac{8}{5} & \frac{1}{5} \end{pmatrix}, \qquad P = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}.$$

Carrying out the multiplication (or verifying by direct computation) yields

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = D.$$

Step 8: Verify diagonalization $P^{-1}AP = D$

Compute $P^{-1}AP$. Using the matrices given:

$$P^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{9}{5} & \frac{2}{5} \\ -\frac{4}{5} & -\frac{8}{5} & \frac{1}{5} \end{pmatrix}, \qquad P = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}.$$

Carrying out the multiplication (or verifying by direct computation) yields

$$P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = D.$$

Thus $A = PDP^{-1}$ and diagonalization is confirmed.

Step 9: Use — powers and Cayley–Hamilton remark

- Since A is diagonalizable, $A^k = PD^kP^{-1}$. That reduces powering A to powering diagonal entries $5^k, 10^k, 10^k$.

Step 9: Use — powers and Cayley–Hamilton remark

- Since A is diagonalizable, $A^k = PD^kP^{-1}$. That reduces powering A to powering diagonal entries $5^k, 10^k, 10^k$.
- Cayley–Hamilton: $p_A(A) = (A - 10I)^2(A - 5I) = 0$. From the cubic relation you can express high powers of A as linear combinations of I, A, A^2 or isolate an explicit formula for A^{-1} in terms of A^2, A, I (since constant term is nonzero).

Summary

- Characteristic polynomial: $p_A(\lambda) = (\lambda - 10)^2(\lambda - 5)$.
- Eigenvalues and AM: $\lambda = 5$ (AM=1), $\lambda = 10$ (AM=2).
- Geometric multiplicities: $\mathrm{GM}(5) = 1$, $\mathrm{GM}(10) = 2$. So GM = AM for each eigenvalue and A is diagonalizable.
- Integer eigenvectors: $v_5 = (5, -2, 4)$, $v_{10}^{(1)} = (-2, 1, 0)$, $v_{10}^{(2)} = (-1, 0, 1)$. Normalized vectors shown separately.
- Explicit modal matrix P, $\det(P) = 5$, adjugate and P^{-1} given; verified $P^{-1}AP = D$.

Optional Topics (Not in syllabus):

- Gram–Schmidt orthonormalisation process.
- Singular Value Decomposition (SVD).

Thank You!

Dr. D Bhanu Prakash
dbhanuprakash233.github.io
Mail: db_maths@vignan.ac.in