25MT103: Linear Algebra

Unit 3: Eigenvalues, Eigenvectors and Diagonalization

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Eigenvalues, Eigenvectors - Tutorial Problems

Syllabus

- Characteristic Equation
- Eigenvalues, Eigenvectors and their Properties
- Cayley-Hamilton Theorem
- Diagonalization of a Matrix (only for diagonalizable matrices)
- Inverse of a matrix by Cayley-Hamilton Theorem
- Power of a diagonalizable square matrix

Eigenvalues and Eigenvectors

Compute eigenvalues and eigenvectors for the following matrices and decide diagonalizability.

1. $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.
2. $B = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
3. $C = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}$.

Hint: Characteristic Polynomial =
$-\lambda^3 + (\mathrm{tr}(A))\lambda^2 - (\text{Sum of minors of principle diagonal})\lambda + (\det(A))$.

Inverse via Cayley–Hamilton (2×2)

Method

If $p_A(\lambda) = \lambda^2 + a_1\lambda + a_0$ and $a_0 \neq 0$, then from $p_A(A) = 0$ we can isolate A^{-1}:

$$A^{-1} = -\frac{1}{a_0}(A + a_1 I).$$

Inverse via Cayley–Hamilton (2×2)

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Example

$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ has $p_A(\lambda) = \lambda^2 - 7\lambda + 10$. So

$$A^2 - 7A + 10I = 0 \Rightarrow A^{-1} = \frac{1}{10}(7I - A).$$

Practice Problems

Problem A (2×2)

Find eigenvalues/vectors of $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, diagonalize it, compute A^{10}.

Problem B (3×3)

For $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ find eigenvalues, multiplicities, and decide diagonalizability.

Problem C (2×2)

Compute inverse of $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ using Cayley–Hamilton.

More Practice Problems

1. For $C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$ compute eigenvalues and decide diagonalizability.
2. Use Cayley–Hamilton to compute A^{-1} for $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.
3. Given a diagonalizable matrix with eigenpairs, show how to compute e^{At} using diagonalization.

Thank You!

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