25MT103: Linear Algebra

Unit 4: Real Vector Space

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Real Vector Space - Lecture Slides

Syllabus

- Real vector space
- Subspace
- Linear dependence and independence
- Linear span
- Bases and dimension
- Row space and column space of a matrix
- Determining rank of a matrix using row space and column space
- Row and column spaces of similar matrices

Outline

Outline

Vectors in a Vector Space

Definition

Let V be a real vector space over the field $\mathbb{R}$. The elements of V are called *vectors*. These may be:

- Ordered tuples of real numbers (e.g. $(x_1, \ldots, x_n) \in \mathbb{R}^n$),
- Functions $f : \mathbb{R} \to \mathbb{R}$ (in function spaces),
- Polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ (in polynomial spaces), or
- Matrices or sequences.

Notation

Vectors are often denoted by boldface letters ($\mathbf{v}$, $\mathbf{u}$) or arrows ($\vec{v}$), and the zero vector is written $\mathbf{0}$.

Outline

Linear Combination of Vectors

Definition

A *linear combination* of vectors $v_1, v_2, \ldots, v_k$ in a vector space V is any vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k,$$

where $\alpha_i \in \mathbb{R}$ are scalars.

Problems: Linear Combination

Problem 1

Express $(3,5)$ as a linear combination of $(1,1)$ and $(1,2)$.

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Solution

We need $a(1,1)+b(1,2)=(3,5)$.

$$a+b=3$$
$$a+2b=5.$$

Subtract first from second: $b=2$, hence $a=1$. So $(3,5)=1(1,1)+2(1,2)$.

Problems: Linear Combination

Problem 2

Check if $(2,4,6)$ is a linear combination of $(1,0,1)$, $(0,1,1)$.

Problems: Linear Combination

Problem 2

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Solution

We need $a(1,0,1)+b(0,1,1)=(2,4,6)$ giving $a=2$, $b=4$, and $a+b=6$ which holds. Hence yes.

Outline

Linear Span

Definition

For a subset $S \subset V$, the *span* of S is

$$\operatorname{span}(S) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid k \in \mathbb{N},\ v_i \in S,\ \alpha_i \in \mathbb{R}\}.$$

It is the smallest subspace containing S.

Problems: Span

Problem 1

Find $\text{span}\{(1,2,3),(2,4,6)\}$ in $\mathbb{R}^3$.

Problems: Span

Problem 1

Find $\text{span}\{(1,2,3),(2,4,6)\}$ in $\mathbb{R}^3$.

Solution

Note $(2,4,6) = 2(1,2,3)$ so the span is all multiples of $(1,2,3)$, i.e. a 1-dimensional subspace $\{t(1,2,3) \mid t \in \mathbb{R}\}$.

Problem 2

Does $\text{span}\{(1,0,1),(0,1,1)\}$ equal $\mathbb{R}^3$?

Problems: Span

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Problem 2

Does $\text{span}\{(1,0,1),(0,1,1)\}$ equal $\mathbb{R}^3$?

Solution

We need to check if an arbitrary (x,y,z) can be written as $a(1,0,1)+b(0,1,1) = (a,b,a+b)$. This yields $x = a$, $y = b$, and $z = a+b = x+y$. So (x,y,z) is in the span iff $z = x+y$. Thus the span is the plane $\{(x,y,z) : z = x+y\}$, not all $\mathbb{R}^3$.

3D Visual

Link

https://trkern.github.io/span3.html

Outline

Linear Dependence / Independence

Definition

A finite set $\{v_1, \ldots, v_k\}$ in V is *linearly dependent* if there exist scalars $\alpha_1, \ldots, \alpha_k$, not all zero, such that

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0.$$

If the only solution is $\alpha_1 = \cdots = \alpha_k = 0$, the set is *linearly independent*.

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If the only solution is $\alpha_1 = \cdots = \alpha_k = 0$, the set is *linearly independent*.

Important properties

- Any set containing the zero vector is linearly dependent.
- Any subset of a linearly independent set is independent.
- If $k > \dim V$, any list of k vectors is dependent.

Problems: Dependence / Independence

Problem 1

Determine if the vectors $(1, 2, 3)$, $(0, 1, 2)$, and $(1, 3, 5)$ in $\mathbb{R}^3$ are linearly independent.

Problems: Dependence / Independence

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Determine if the vectors $(1,2,3)$, $(0,1,2)$, and $(1,3,5)$ in $\mathbb{R}^3$ are linearly independent.

Solution

Solve $a(1,2,3)+b(0,1,2)+c(1,3,5)=(0,0,0)$. This gives the system:

$$\begin{aligned} a+0b+c&=0\\ 2a+b+3c&=0\\ 3a+2b+5c&=0. \end{aligned}$$

From first: $c=-a$. Substitute into second: $2a+b+3(-a)=2a+b-3a$ $=-a+b=0 \Rightarrow b=a$. Third: $3a+2b+5c=3a+2a+5(-a)=0$ — always satisfied. So $b=a, c=-a$; nontrivial solution exists (take $a=1$ gives $(1,1,-1)$). Thus dependent.

Problems: Dependence / Independence

Problem 2

Show that the set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent in $\mathbb{R}^3$.

Problems: Dependence / Independence

Problem 2

Show that the set $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent in $\mathbb{R}^3$.

Solution

Solve $a(1,0,0)+b(0,1,0)+c(0,0,1)=0$. This yields $a=b=c=0$. Hence independent.

Question 1

Does any collection of vectors become a vector space?

Question 2

What is the difference between a set and a space?

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What is the difference between a set and a space?

Solution

- A **set** is simply a collection of distinct elements with no additional structure.
- A **space** is a set *with structure* — meaning we define extra operations or relations on it.

Examples:

- Set: $\{1,2,3,4\}$ (no structure)
- Vector Space: $\mathbb{R}^2$ under usual addition and scalar multiplication
- Metric Space: $(\mathbb{R}, d)$ where $d(x,y) = |x-y|$

Conclusion: Every space is a set, but not every set is a space.

Question 1

Does any collection of vectors become a vector space?

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Does any collection of vectors become a vector space?

Solution

No. Not every collection of vectors forms a vector space. To be a vector space, a set must satisfy all the **vector space axioms**, including:

- Closure under addition and scalar multiplication
- Existence of a zero vector
- Existence of additive inverses
- Associativity and commutativity of addition
- Distributive and scalar associative laws

If even one property fails, the collection is **not** a vector space.

Outline

Real Vector Space

Definition

A *real vector space* V is a set together with two operations: vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$, such that the following axioms hold for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:

1. $u + v = v + u$ (commutativity),
2. $(u + v) + w = u + (v + w)$ (associativity),
3. There exists $0 \in V$ with $v + 0 = v$ for all v (Additive identity)
4. each v has an additive inverse $-v$,
5. $\alpha(v + w) = \alpha v + \alpha w$ (Distributiivity)
6. $(\alpha + \beta)v = \alpha v + \beta v$ (Distributiivity)
7. $(\alpha\beta)v = \alpha(\beta v)$
8. $1 \cdot v = v$.

Important properties

- The zero vector is unique.
- $0 \cdot v = \mathbf{0}$ and $\alpha \cdot \mathbf{0} = \mathbf{0}$ for all α.
- $(-1)v = -v$.

Example

$\mathbb{R}^n$ with usual addition and scalar multiplication.

Example

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Solution

- Vectors in $\mathbb{R}^n$ can be added:
  $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$
- Scalars multiply vectors: $c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$
- All vector space axioms hold (closure, zero vector, inverses, etc.)

Hence, $\mathbb{R}^n$ is the most familiar example of a vector space.

Counter Example

Give an example of a set that is *not* a vector space.

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Set

Let $S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

Counter Example

Give an example of a set that is *not* a vector space.

Set

Let $S = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$

Solution

- S is closed under addition.
- But not closed under scalar multiplication: for scalar -1, $-1 \cdot (1, 1) = (-1, -1) \notin S$.

Hence, S is not a vector space.

Example

Can the set of all polynomials with usual addition and scalar multiplication be a vector space?

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Solution

The set of all real polynomials, denoted $\mathscr{P} = \{a_0 + a_1 x + a_2 x^2 + \dots\}$, is a vector space over $\mathbb{R}$ because:

- Sum of two polynomials is a polynomial.
- Scalar multiple of a polynomial is a polynomial.
- There exists a zero polynomial (0).

Thus, $\mathscr{P}$ satisfies all vector space axioms.

Example

Give a counterexample — a set of polynomials that is *not* a vector space.

Example

Give a counterexample — a set of polynomials that is *not* a vector space.

Solution

Let $S = \{p(x) \in \mathscr{P} : p(0) = 1\}$

- For $p(x), q(x) \in S$, $(p+q)(0) = p(0) + q(0) = 2 \neq 1$
- So S is **not closed** under addition

Hence, S is **not** a vector space.

Example

Verify that the set $W = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ is a real vector space with pointwise addition and scalar multiplication.

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Verify that the set $W = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ is a real vector space with pointwise addition and scalar multiplication.

Solution

1. Sum of continuous functions is continuous (standard result). Hence closed under addition.
2. Scalar multiple of continuous function is continuous. Hence closed under scalar multiplication.
3. All axioms (associativity, distributivity, etc.) hold because they hold for real-valued functions pointwise. Therefore W is a real vector space.

Examples of Vector Spaces:

- $\mathbb{R}^n$, $\mathbb{C}^n$
- All polynomials
- All continuous functions $C[a,b]$

Counterexamples:

- $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$
- $\{p(x) : p(0) = 1\}$

Summary

How do we check if something is a vector space?

Answer: Verify the following 10 axioms (for addition and scalar multiplication):

1. Closure under addition
2. Commutativity of addition
3. Associativity of addition
4. Existence of additive identity
5. Existence of additive inverse
6. Closure under scalar multiplication
7. Distributive property (scalar over vectors)
8. Distributive property (vector over scalars)
9. Compatibility of scalar multiplication
10. Existence of multiplicative identity

If all hold — it's a vector space!

Outline

Subspace

Definition

A subset U of a vector space V is a *subspace* if U is itself a vector space under the operations inherited from V. Equivalently, U is nonempty and closed under addition and scalar multiplication.

Important properties

- Intersection of subspaces is a subspace.
- Sum $U+W=\{u+w \mid u\in U, w\in W\}$ is the smallest subspace containing $U\cup W$.

Problems: Subspaces

Problem 1

Let $U = \{(x, x, 0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3$. Show U is a subspace.

Problems: Subspaces

Problem 1

Let $U = \{(x,x,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3$. Show U is a subspace.

Solution

1. Nonempty: $(0,0,0) \in U$.
2. Closed under addition: $(x,x,0)+(y,y,0) = (x+y,x+y,0) \in U$.
3. Closed under scalars: $\alpha(x,x,0) = (\alpha x, \alpha x, 0) \in U$.
4. So U is a subspace.

Problems: Subspaces

Problem 2

Is the set $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ a subspace of $\mathbb{R}^3$?

Problems: Subspaces

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Is the set $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ a subspace of $\mathbb{R}^3$?

Solution

This set is the unit sphere. It does not contain the zero vector, and it's not closed under scalar multiplication (e.g. $2 \cdot (1,0,0) = (2,0,0)$ not in S). Hence not a subspace.

Problem

Let $V = \{(x, y, z) \in \mathbb{R}^3 : \ x + y + z = 0\}$. Show V is a real vector space (subspace of $\mathbb{R}^3$).

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Let $V = \{(x,y,z) \in \mathbb{R}^3 : \; x+y+z=0\}$. Show V is a real vector space (subspace of $\mathbb{R}^3$).

Solution

1. Check closure under addition: take $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ with sums zero. Then

   $$(x_1+x_2)+(y_1+y_2)+(z_1+z_2) = (x_1+y_1+z_1)+(x_2+y_2+z_2) = 0+0 = 0,$$

   so $u+v \in V$.
2. Check closure under scalar multiplication: for $\alpha \in \mathbb{R}$,

   $$\alpha x + \alpha y + \alpha z = \alpha(x+y+z) = \alpha \cdot 0 = 0,$$

   so $\alpha v \in V$.
3. Nonempty: $(1,-1,0) \in V$, so $V \neq \emptyset$. Thus V is a subspace and hence a real vector space.

Outline

Bases and Dimension

Definition

A *basis* of a vector space V is a list of vectors that is linearly independent and spans V. The *dimension* of V, $\dim V$, is the number of vectors in any basis (if finite).

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A *basis* of a vector space V is a list of vectors that is linearly independent and spans V. The *dimension* of V, $\dim V$, is the number of vectors in any basis (if finite).

Important properties

- Every finite-dimensional vector space has a basis.
- All bases of a finite-dimensional vector space have the same number of elements.
- If $\dim V = n$, any list of n independent vectors is a basis; any spanning list with n vectors is a basis.

Problems: Bases and Dimension

Problem 1

Find a basis and dimension of $V = \{(x,y,z) \in \mathbb{R}^3 \mid x+2y+3z=0\}$.

Problems: Bases and Dimension

Problem 1

Find a basis and dimension of $V = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$.

Solution

Solve $x = -2y - 3z$. Then vectors are $(x,y,z) = y(-2,1,0) + z(-3,0,1)$. So $V = \text{span}\{(-2,1,0),(-3,0,1)\}$. Check independence: only trivial combination gives zero (quick determinant or reasoning), hence they form a basis. Dimension $= 2$.

Problems: Bases and Dimension

Problem 2

Show that any two bases of a finite-dimensional vector space have the same number of elements.

Problems: Bases and Dimension

Problem 2

Show that any two bases of a finite-dimensional vector space have the same number of elements.

Solution

Let B and B' be bases. Since B' spans, each vector of B is a linear combination of B'. Using the exchange lemma one shows $|B| \leq |B'|$. By symmetry $|B'| \leq |B|$. So $|B| = |B'|$.

Outline

Row space and Column space

Definitions

For a matrix $A \in \mathbb{R}^{m\times n}$:

- The *row space* $\mathscr{R}(A)$ is the subspace of $\mathbb{R}^n$ spanned by the rows of A (viewed as row vectors).
- The *column space* $\mathscr{C}(A)$ is the subspace of $\mathbb{R}^m$ spanned by the columns of A.
- The *rank* of A, $\operatorname{rank}(A)$, is $\dim \mathscr{R}(A) = \dim \mathscr{C}(A)$.

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- The *rank* of A, $\operatorname{rank}(A)$, is $\dim \mathscr{R}(A) = \dim \mathscr{C}(A)$.

Important properties

- Row reductions (elementary row operations) do not change the row space (they change which rows generate it) or the row rank; column operations change column space.
- Row rank equals column rank (fundamental theorem of linear algebra).

Problems: Row/Column Space and Rank

Problem 1

Find the row space, column space and rank of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution

Row-reduce A:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2-2R_1, R_3-R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{pmatrix}$$

Problems: Row/Column Space and Rank

Solution

Swap rows $R_2 \leftrightarrow R_3$ and scale $R_2 \to -R_2$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Pivots in columns 1 and 2. Rank $= 2$.
- Row space: span of pivot rows $\{(1,2,3),(0,1,2)\}$.
- Column space: span of $\{c1 = (1,2,1), c2 = (2,4,1), c3 = (3,6,1)\}$.

Problems: Row/Column Space and Rank

Problem 2

Compute rank of $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 7 \end{pmatrix}$.

Problems: Row/Column Space and Rank

Problem 2

Compute rank of $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 7 \end{pmatrix}$.

Solution

Row reduce:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 3 & 7 \end{pmatrix} \xrightarrow{R_3 - 2R_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 3 & 3 \end{pmatrix} \xrightarrow{R_3 - 3R_2} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -6 \end{pmatrix}.$$

All three pivots; rank $= 3$.

Row and Column Spaces of Similar Matrices

Statement

If A and B are similar matrices ($B = P^{-1}AP$ for invertible P), then $\text{rank}(A) = \text{rank}(B)$ and the row/column spaces are isomorphic via the change of basis given by P.

Important properties

- Similarity represents the same linear operator in different bases, so invariant properties like rank, determinant, trace, eigenvalues (multiset) are preserved.

Problem: Similar Matrices

Problem 1

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute $B = P^{-1}AP$ and compare ranks and column spaces.

Problem: Similar Matrices

Problem 1

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Compute $B = P^{-1}AP$ and compare ranks and column spaces.

Solution

Compute $P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then

$$B = P^{-1}AP = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A.$$

Here $B = A$, ranks and column spaces identical. (This choice of P centralizes A.)

Problem: Similar Matrices

Problem 2

Explain why rank is invariant under similarity.

Problem: Similar Matrices

Problem 2

Explain why rank is invariant under similarity.

Solution

$B = P^{-1}AP$. Multiplication by invertible matrices on left/right corresponds to applying invertible linear maps to row/column spaces, which preserves dimensions. Hence rank preserved.

Outline

Summary and Takeaways

- Vector spaces, subspaces: the foundational language.
- Linear dependence/independence, span, bases, dimension — tightly interconnected.
- Row/column spaces and rank are central in matrix theory; rank preserved under similarity.

Thank You!

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