25MT103: Linear Algebra

Unit 5: Real Inner Product Space

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Real Inner Product Space - Lecture Slides

Syllabus

- Real inner product space
- Norm of a vector
- Orthogonal set
- Orthonormal set
- Cauchy-Schwarz inequality
- Gram-Schmidt orthogonalization process.
- QR decomposition
- Singular Value Decomposition (SVD).

Outline

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Real Inner Product Space

Definition

A *real inner product space* is a real vector space V with an operation $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \langle v, u \rangle$ (symmetry)
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (linearity in first argument)
3. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

Properties

- $\|v\| = \sqrt{\langle v, v \rangle}$ defines a norm.
- $\langle u, v \rangle = \|u\| \|v\| \cos \theta$ defines angle between u and v.

Problems: Inner Product

Problem 1

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Solution

Standard inner product: $\langle u,v\rangle = u_1v_1+u_2v_2+u_3v_3$. Thus:

$$1\cdot 4+2\cdot 5+3\cdot 6 = 4+10+18 = 32.$$

Problem 2

Check if $(1,2,3)$ and $(2,4,6)$ are orthogonal in $\mathbb{R}^3$.

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Check if $(1,2,3)$ and $(2,4,6)$ are orthogonal in $\mathbb{R}^3$.

Solution

Compute $\langle(1,2,3),(2,4,6)\rangle = 1\cdot 2 + 2\cdot 4 + 3\cdot 6 = 2 + 8 + 18 = 28 \neq 0$.
Hence not orthogonal.

Outline

Norm, Orthogonal and Orthonormal Sets

Definitions

- The *norm* of v is $\|v\| = \sqrt{\langle v, v\rangle}$.
- Vectors u, v are *orthogonal* if $\langle u, v\rangle = 0$.
- A set $\{v_1, \ldots, v_k\}$ is *orthogonal* if all distinct pairs are orthogonal.
- It is *orthonormal* if it is orthogonal and each $\|v_i\| = 1$.

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Properties

- An orthogonal set of nonzero vectors is linearly independent.
- Any vector in the span of an orthonormal set can be expressed as $v = \sum \langle v, v_i\rangle v_i$.

Problems: Orthogonality and Norm

Problem 1

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$\|v\| = \sqrt{3^2 + 4^2} = 5$.

Problem 2

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Problem 2

Check if $(1,-1,0)$, $(1,1,0)$ are orthogonal.

Solution

Inner product $= 1\cdot 1 + (-1)\cdot 1 + 0\cdot 0 = 0$. Thus orthogonal.

Outline

Cauchy-Schwarz Inequality

Statement

For all u, v in a real inner product space,

$$|\langle u, v\rangle| \leq \|u\| \, \|v\|,$$

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Proof sketch

Consider $f(t) = \|u - tv\|^2 \geq 0$ for all t. Expand:

$$f(t) = \langle u, u\rangle - 2t\langle u, v\rangle + t^2\langle v, v\rangle.$$

The discriminant ≤ 0 gives the inequality.

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Solution

$\langle u, v\rangle = 1\cdot 2 + 2\cdot 1 = 4$, $\|u\| = \sqrt{5}$, $\|v\| = \sqrt{5}$. So $|\langle u, v\rangle| = 4 \leq 5 = \|u\|\|v\|$. Verified.

Problem 2

When does equality hold?

Problems: Cauchy-Schwarz

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Problem 2

When does equality hold?

Solution

Equality iff one vector is scalar multiple of the other, i.e. $u = \lambda v$.

Outline

Gram-Schmidt Process

Procedure

Given linearly independent vectors $v_1, \ldots, v_n$ in inner product space V, produce orthonormal set $u_1, \ldots, u_n$:

$$u_1 = \frac{v_1}{\|v_1\|},$$

$$u_2' = v_2 - \langle v_2, u_1 \rangle u_1, \quad u_2 = \frac{u_2'}{\|u_2'\|},$$

$$u_k' = v_k - \sum_{j=1}^{k-1} \langle v_k, u_j \rangle u_j, \quad u_k = \frac{u_k'}{\|u_k'\|}.$$

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Properties

- Output set is orthonormal and spans same subspace as input.
- Stable under scaling and order of orthogonalization.

Problems: Gram-Schmidt

Problem 1

Apply Gram-Schmidt to $(1,1,0)$ and $(1,0,1)$.

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Solution

Let $v_1 = (1,1,0)$, $v_2 = (1,0,1)$.

$$u_1 = \frac{(1,1,0)}{\sqrt{2}},$$

$$u_2' = (1,0,1) - \langle (1,0,1), u_1 \rangle u_1 = (1,0,1) - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,1,0) = (1/2, -1/2, 1).$$

$$u_2 = \frac{u_2'}{\|u_2'\|} = \frac{(1/2,-1/2,1)}{\sqrt{(1/2)^2 + (-1/2)^2 + 1^2}} = \frac{(1/2,-1/2,1)}{\sqrt{3/2}}.$$

Outline

QR Decomposition

Definition

Any real matrix $A \in \mathbb{R}^{m\times n}$ with linearly independent columns can be decomposed as

$$A = QR,$$

where Q has orthonormal columns ($Q^T Q = I$) and R is upper triangular. This is the *QR decomposition*.

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Computation

Gram-Schmidt on columns of A gives Q; $R = Q^T A$.

Problems: QR Decomposition

Problem

Find the QR decomposition of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$.

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Solution

Columns: $a_1 = (1,0,1)$, $a_2 = (1,1,0)$.

$$\begin{aligned} q_1 &= a_1/\|a_1\| = (1,0,1)/\sqrt{2}, \\ \mathrm{proj}_{q_1}(a_2) &= \langle a_2, q_1\rangle q_1 = (1\cdot 1 + 1\cdot 0 + 0\cdot 1)/\sqrt{2}\cdot(1,0,1)/\sqrt{2} = (1/2)(1,0,1), \\ u_2' &= a_2 - \mathrm{proj}_{q_1}(a_2) = (1,1,0) - (1/2,0,1/2) = (1/2,1,-1/2), \\ q_2 &= u_2'/\|u_2'\|. \\ R &= Q^T A. \end{aligned}$$

Singular Value Decomposition (SVD)

Definition

For any $A \in \mathbb{R}^{m\times n}$, there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$, and a diagonal matrix Σ with nonnegative entries (singular values) such that

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Properties

- Columns of U are eigenvectors of AA^T; columns of V are eigenvectors of $A^T A$.
- Singular values σ_i are square roots of eigenvalues of $A^T A$.

Problems: SVD

Problem 1

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Solution

Compute $A^T A = \begin{pmatrix} 9 & 0 \\ 0 & 16 \end{pmatrix}$. Eigenvalues $= 9, 16$. Singular values are $\sqrt{9} = 3$, $\sqrt{16} = 4$.

Problem 2

Explain significance of SVD.

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Solution

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Problem 2

Explain significance of SVD.

Solution

SVD expresses linear transformations as rotations/reflections and scalings. Useful in PCA, signal compression, and least squares.

Summary

- Linear combination, inner product, and orthogonality form the core of Euclidean geometry in vector spaces.
- Gram-Schmidt leads to QR decomposition.
- SVD generalizes orthogonal diagonalization for all matrices.

Thank You!

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