SRI SATHYA SAI INSTITUTE OF HIGHER LEARNING
(Deemed to be University)
Department of Mathematics and Computer Science

TUTORIAL 1 - SOLUTIONS

Mathematical Review & Unconstrained Optimization

UMAT-503: Optimization Techniques	Instructor: Dr. D Bhanu Prakash	Date: 20-07-2026

Q1. Convex sets, neighbourhoods, and polyhedra

(a) S1 is convex:

$S1 = \{x \in R^2 : x1 + 2x2 \leq 4, x1 \geq 0, x2 \geq 0\}$

Let u, v $\in$ S1 and $\lambda \in [0,1]$. Consider $w = \lambda u + (1-\lambda)v$.

Since $u1 + 2u2 \leq 4$ and $v1 + 2v2 \leq 4$:

$w1 + 2w2 = \lambda(u1+2u2) + (1-\lambda)(v1+2v2) \leq \lambda(4) + (1-\lambda)(4) = 4$

Similarly $w1 = \lambda u1+(1-\lambda)v1 \geq 0$ and $w2 \geq 0$ since all terms are non-negative.

Hence w $\in$ S1, so S1 is convex (in general, any set defined by a system of linear inequalities — a half-space intersection — is always convex).

(b) S2 is NOT convex:

$S2 = \{x \in R^2 : x1^2 + x2^2 \geq 1\}$ (the region outside/on the unit circle).

Counterexample: take $u = (-1, 0)$ and $v = (1, 0)$, both in S2 since $1 \geq 1$.

Midpoint $w = (0,0)$: $w1^2+w2^2 = 0$, which is NOT ≥ 1.

So w $\notin$ S2, even though u, v $\in$ S2. Hence S2 is not convex.

(c) ε-neighbourhood and polyhedron representation:

The ε-neighbourhood of a point $x0 \in R^n$ is the set $N\varepsilon(x0) = \{x \in R^n : \|x - x0\| < \varepsilon\}$, i.e. all points within Euclidean distance ε of x0 (an open ball of radius ε).

S1 IS a polyhedron, since it is the intersection of finitely many half-spaces:

$S1 = \{x : x1 + 2x2 \leq 4\} \cap \{x : -x1 \leq 0\} \cap \{x : -x2 \leq 0\}$

i.e. three half-planes bounded by the hyperplanes x1+2x2=4, x1=0, and x2=0, forming a triangular polyhedral region.

Q2. Hyperplane through three points

Given P1=(1,0,1), P2=(0,1,1), P3=(1,1,0).

Direction vectors in the plane:

$v1 = P2 - P1 = (-1, 1, 0)$
$v2 = P3 - P1 = (0, 1, -1)$

Normal vector (cross product) $n = v1 \times v2$:
$n_x = (1)(-1) - (0)(1) = -1$
$n_y = (0)(0) - (-1)(-1) = -1$
$n_z = (-1)(1) - (1)(0) = -1$
$n = (-1, -1, -1) \Rightarrow$ simplify to $n = (1, 1, 1)$

Plane equation: using point $P1 = (1,0,1)$:
$1(x-1) + 1(y-0) + 1(z-1) = 0 \Rightarrow x + y + z = 2$

Check: P2: 0+1+1=2 ✓ P3: 1+1+0=2 ✓

Linear-variety form: $\{x \in R^3 : a^T x = b\}$ with $a = (1,1,1)^T$ and $b = 2$.

Q3. Gradient and level sets

$f(x,y) = x^2 + 2y^2$

Gradient: $\nabla f = (2x, 4y)$. At (1,1): $\nabla f(1,1) = (2, 4)$

Level curve through (1,1): $f(1,1) = 1 + 2 = 3$, so the level curve is $x^2 + 2y^2 = 3$.

Tangent direction of the level curve (implicit differentiation):
$2x\,dx + 4y\,dy = 0 \Rightarrow dy/dx = -x/(2y)$. At (1,1): $dy/dx = -1/2$
Tangent direction vector: $(2, -1)$ (using slope $-1/2$ scaled to integers).

Orthogonality check:
$\nabla f(1,1) \cdot (\text{tangent direction}) = (2,4)\cdot(2,-1) = 4 - 4 = 0$

Since the dot product is zero, the gradient is perpendicular to the tangent of the level curve at (1,1) — confirming the general property that ∇f is always normal to the level set of f at any point.

Q4. Taylor series expansion

$f(x,y) = e^x \ln(1+y)$, expand about (0,0) up to second order.

Function value: $f(0,0) = e^0 \ln(1) = 0$

First partial derivatives:

$f_x = e^x \ln(1+y) \;\Rightarrow\; f_x(0,0) = 0$

$f_y = e^x/(1+y) \;\Rightarrow\; f_y(0,0) = 1$

Second partial derivatives:

$f_{xx} = e^x \ln(1+y) \;\Rightarrow\; f_{xx}(0,0) = 0$

$f_{xy} = e^x/(1+y) \;\Rightarrow\; f_{xy}(0,0) = 1$

$f_{yy} = -e^x/(1+y)^2 \;\Rightarrow\; f_{yy}(0,0) = -1$

Second-order Taylor expansion about (0,0):

$$f(x,y) \approx f(0,0) + f_x\cdot x + f_y\cdot y + (1/2)[f_{xx}\cdot x^2 + 2f_{xy}\cdot xy + f_{yy}\cdot y^2]$$

$$= 0 + 0 + y + (1/2)[0 + 2xy - y^2]$$

$$= y + xy - y^2/2$$

Q5. Conditions for local minimizers

$f(x,y) = x^3 + y^3 - 3xy$

First-order necessary condition ($\nabla f = 0$):

$f_x = 3x^2 - 3y = 0 \;\Rightarrow\; y = x^2$

$f_y = 3y^2 - 3x = 0 \;\Rightarrow\; x = y^2$

Substituting $y = x^2$ into $x = y^2$: $x = x^4 \;\Rightarrow\; x^4 - x = 0 \;\Rightarrow\; x(x^3 - 1) = 0$

So $x = 0$ or $x = 1$.

$x=0 \Rightarrow y=0$. $x=1 \Rightarrow y=1$.

Critical points: (0,0) and (1,1).

Second-order (Hessian) sufficient condition:

$H(x,y) = [\,[f_{xx}, f_{xy}], [f_{yx}, f_{yy}]\,] = [\,[6x, -3], [-3, 6y]\,]$

At (0,0): $H = [\,[0,-3], [-3,0]\,]$, $\det(H) = (0)(0) - (-3)(-3) = -9 < 0$

$\Rightarrow$ Indefinite Hessian $\Rightarrow$ (0,0) is a SADDLE POINT.

At (1,1): $H = [\,[6,-3], [-3,6]\,]$, $\det(H) = 36 - 9 = 27 > 0$, and $f_{xx} = 6 > 0$

$\Rightarrow$ Positive definite Hessian $\Rightarrow$ (1,1) is a LOCAL MINIMIZER.

Q6. Golden section search and Fibonacci search

(a) Golden section search on $f(x) = x^2 + 2x$ over $[a,b] = [-3, 2]$:

Golden ratio factor $\tau = 0.618$, so $1-\tau = 0.382$. Interval length $L = b-a = 5$.

Iteration 1:

$x1 = a + 0.382L = -3 + 1.910 = -1.09$

$x2 = a + 0.618L = -3 + 3.090 = 0.09$

$f(x1) = (-1.09)^2 + 2(-1.09) = 1.1881 - 2.18 = -0.9919$

$f(x2) = (0.09)^2 + 2(0.09) = 0.0081 + 0.18 = 0.1881$

Since $f(x1) < f(x2)$, the minimum lies in $[a, x2]$; new interval = $[-3, 0.09]$ (length 3.09).

Iteration 2:

Reuse $x1 = -1.09$ as the new right test point $x2'$.

New left test point: $x1' = a + 0.382(3.09) = -3 + 1.1804 = -1.8196$

$f(x1') = (-1.8196)^2 + 2(-1.8196) = 3.311 - 3.639 = -0.328$

$f(x2') = f(-1.09) = -0.9919$ (already computed)

Since $f(x1') > f(x2')$, the minimum lies in $[x1', b]$; new interval = $[-1.8196, 0.09]$ (length $1.9096 \approx 5 \times 0.382^2$ ✓).

After 2 iterations, the interval of uncertainty has shrunk from length 5 to about 1.91, bracketing the true minimizer $x^* = -1$ ($f(-1) = -1$).

(b) Fibonacci search vs golden section search:

Fibonacci search uses ratios based on the Fibonacci sequence (F_n), and the reduction factor at each iteration VARIES — it depends on how many iterations n remain, and requires the total number of iterations to be fixed in advance. This makes Fibonacci search the theoretically OPTIMAL method (minimum worst-case final interval width for a given number of function evaluations). Golden section search is the limiting case of Fibonacci search as $n \to \infty$, using the constant ratio 0.618 at every iteration. Its advantage is that it does not require knowing n in advance and is simpler to implement, since the reduction ratio is always the same at each step — at a very slight loss of optimality compared to Fibonacci search.

Q7. Bisection method vs Newton's method

$f(x) = e^x - 2x$, $f'(x) = e^x - 2$, $f''(x) = e^x$

The minimizer satisfies $f'(x)=0 \Rightarrow x^* = \ln 2 \approx 0.6931$

(a) Bisection method applied to $f'(x)=0$ on [0,1]:

$f'(0) = 1 - 2 = -1 < 0$; $f'(1) = 2.71828 - 2 = 0.71828 > 0$ (sign change confirmed)

Iteration 1: mid = 0.5, $f'(0.5) = 1.6487 - 2 = -0.3513 < 0 \Rightarrow$ new interval [0.5, 1]

Iteration 2: mid = 0.75, $f'(0.75) = 2.117 - 2 = 0.117 > 0 \Rightarrow$ new interval [0.5, 0.75]

Iteration 3: mid = 0.625, $f'(0.625) = 1.868 - 2 = -0.132 < 0 \Rightarrow$ new interval [0.625, 0.75]

After 3 iterations: interval = [0.625, 0.75], midpoint estimate ≈ 0.6875 (true $x^* \approx 0.6931$).

(b) Newton's method starting at x0 = 0:

Update rule: $x_{k+1} = x_k - f'(x_k)/f''(x_k)$

$x1 = 0 - (e^0 - 2)/e^0 = 0 - (1-2)/1 = 1$

$x2 = 1 - (e^1 - 2)/e^1 = 1 - (2.71828-2)/2.71828 = 1 - 0.2643 = 0.7357$

After 2 iterations: x2 = 0.7357, already close to the true value 0.6931.

(c) Convergence comparison:

Newton's method converges QUADRATICALLY near the solution (error roughly squares each step) because it uses curvature information (f″); it reached 0.7357 in just 2 steps versus bisection's [0.625,0.75] bracket after 3 steps. However, Newton's method requires both f′ and f″ to exist and be computable, and can diverge or behave erratically if f″ is not positive (not locally convex) or if the initial guess is far from the minimizer. Bisection converges only LINEARLY (the interval halves every iteration, roughly one bit of accuracy per step) but is much more robust — it only needs the sign of f′ and is guaranteed to converge given a valid bracketing interval, regardless of the function's curvature.