Optimality of the Fibonacci Search Method

A fully detailed reconstruction of the proof

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1 Conventions

Throughout, the Fibonacci numbers F_n are indexed so that

$$F_{-1} = 0, \qquad F_0 = 1, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1} \quad (n \geq 0).$$

n	-1	0	1	2	3	4	5	6	7
F_n	0	1	1	2	3	5	8	13	21

All F_n with $n \geq -1$ are nonnegative, and $F_n > 0$ for $n \geq 0$.

2 From the ρ-problem to the r-problem

The Fibonacci search method chooses reduction ratios $\rho_1, \ldots, \rho_N$. The claim to be proved is that this particular choice solves

$$\begin{aligned} \text{minimize} \quad & (1-\rho_1)(1-\rho_2)\cdots(1-\rho_N) \\ \text{subject to} \quad & \rho_{k+1} = 1 - \frac{\rho_k}{1-\rho_k}, \qquad k = 1, \ldots, N-1, \\ & 0 \leq \rho_k \leq \tfrac{1}{2}, \qquad k = 1, \ldots, N. \end{aligned} \tag{P$_\rho$}$$

Let $r_k := 1 - \rho_k$. Then

$$0 \leq \rho_k \leq \tfrac{1}{2} \iff 0 \leq 1 - r_k \leq \tfrac{1}{2} \iff \tfrac{1}{2} \leq r_k \leq 1.$$

Also, substitute $\rho_k = 1 - r_k$ into $\rho_{k+1} = 1 - \dfrac{\rho_k}{1-\rho_k}$:

$$1 - r_{k+1} = 1 - \frac{1 - r_k}{1 - (1 - r_k)} = 1 - \frac{1 - r_k}{r_k}.$$

Cancel the leading 1's from both sides and multiply by -1:

$$r_{k+1} = \frac{1-r_k}{r_k} = \frac{1}{r_k} - 1.$$

Thus (P_ρ) is equivalent to

$$\begin{aligned} &\text{minimize} && r_1 r_2 \cdots r_N \\ &\text{subject to} && r_{k+1} = \frac{1}{r_k} - 1, \qquad k = 1, \dots, N-1, \\ &&& \tfrac{1}{2} \le r_k \le 1, \qquad k = 1, \dots, N. \end{aligned} \tag{P$_r$}$$

All work below is done on (P_r).

The constraint $r_k \le 1$ can be dropped because it is already forced by $r_k \ge \frac{1}{2}$ together with the recursion.

From now on we work with sequences $r_1, r_2, \dots$ satisfying

$$r_{k+1} = \frac{1}{r_k} - 1, \qquad r_k \ge \frac{1}{2}, \qquad k = 1, 2, \dots. \tag{$**$}$$

3 Three technical lemmas

Lemma 1. *For every $k \ge 2$,*

$$r_k = -\frac{F_{k-2} - F_{k-1} r_1}{F_{k-3} - F_{k-2} r_1}.$$

Proof. Base case $k = 2$. Using $F_{-1} = 0,\ F_0 = 1,\ F_1 = 1$,

$$-\frac{F_0 - F_1 r_1}{F_{-1} - F_0 r_1} = -\frac{1 - r_1}{0 - r_1} = -\frac{1 - r_1}{-r_1} = \frac{1 - r_1}{r_1} = \frac{1}{r_1} - 1 = r_2,$$

using $(**)$ in the last step. So the formula holds at $k = 2$.

Inductive step. Suppose the formula holds for some $k \ge 2$; we prove it for $k+1$. Write, for brevity, $a = F_{k-2} - F_{k-1} r_1$ and $b = F_{k-3} - F_{k-2} r_1$, so that the induction hypothesis reads $r_k = -a/b$. Taking the reciprocal,

$$\frac{1}{r_k} = -\frac{b}{a}.$$

By $(**)$,

$$r_{k+1} = \frac{1}{r_k} - 1 = -\frac{b}{a} - 1 = \frac{-b-a}{a} = -\frac{a+b}{a}.$$

Now

$$a + b = (F_{k-2} - F_{k-1} r_1) + (F_{k-3} - F_{k-2} r_1) = (F_{k-2} + F_{k-3}) - (F_{k-1} + F_{k-2}) r_1.$$

By the Fibonacci recurrence, $F_{k-2}+F_{k-3}=F_{k-1}$ and $F_{k-1}+F_{k-2}=F_k$, so

$$a+b=F_{k-1}-F_k r_1.$$

Hence

$$r_{k+1}=-\,\frac{F_{k-1}-F_k r_1}{a}=-\,\frac{F_{k-1}-F_k r_1}{F_{k-2}-F_{k-1}r_1},$$

which is exactly the asserted formula with k replaced by $k+1$ (check: $(k+1)-2=k-1$, $(k+1)-1=k$, $(k+1)-3=k-2$, $(k+1)-2=k-1$). This completes the induction. $\square$

Lemma 2. *For every $k\ge 2$,*

$$(-1)^k\big(F_{k-2}-F_{k-1}r_1\big)>0.$$

Proof. Base case $k=2$. We must show $F_0-F_1r_1=1-r_1>0$, i.e. $r_1<1$. By $(**)$, $r_2\ge\frac{1}{2}$; applying $r_2=\frac{1}{r_1}-1$ gives $r_1\le\frac{2}{3}<1$.

Inductive step. Suppose the claim holds for some $k\ge 2$, i.e. $(-1)^k(F_{k-2}-F_{k-1}r_1)>0$; we prove it for $k+1$. By Lemma 1 applied at index $k+1$ (legitimate since $k+1\ge 3\ge 2$),

$$r_{k+1}=-\,\frac{F_{k-1}-F_k r_1}{F_{k-2}-F_{k-1}r_1},$$

so, solving for the numerator,

$$F_{k-1}-F_k r_1=-\,r_{k+1}\big(F_{k-2}-F_{k-1}r_1\big).$$

Multiply both sides by $(-1)^{k+1}$:

$$(-1)^{k+1}\big(F_{k-1}-F_k r_1\big)=(-1)^{k+1}\cdot(-1)\,r_{k+1}\big(F_{k-2}-F_{k-1}r_1\big)=(-1)^k r_{k+1}\big(F_{k-2}-F_{k-1}r_1\big),$$

because $(-1)^{k+1}\cdot(-1)=(-1)^{k+2}=(-1)^k$. Regroup the right-hand side as

$$(-1)^{k+1}\big(F_{k-1}-F_k r_1\big)=r_{k+1}\cdot\Big[(-1)^k\big(F_{k-2}-F_{k-1}r_1\big)\Big].$$

The bracketed factor is positive by the inductive hypothesis, and $r_{k+1}\ge\frac{1}{2}>0$ by $(**)$. A product of two positive numbers is positive, so

$$(-1)^{k+1}\big(F_{k-1}-F_k r_1\big)>0,$$

which is the claim for $k+1$ (note $(k+1)-2=k-1$ and $(k+1)-1=k$). This completes the induction. $\square$

Lemma 3. *For every $k\ge 2$,*

$$(-1)^{k+1}r_1\ \ge\ (-1)^{k+1}\frac{F_k}{F_{k+1}}.$$

Proof. Since $r_k \geq \frac{1}{2}$, $r_{k+1} = \frac{1}{r_k} - 1$ gives $r_{k+1} \leq 1$. Now apply Lemma 1 at index $k+1 \geq 3$ to rewrite r_{k+1}:

$$-\frac{F_{k-1} - F_k r_1}{F_{k-2} - F_{k-1} r_1} \;\leq\; 1. \tag{1}$$

Multiply numerator and denominator of the left side by $(-1)^k$ (this changes neither the value of the fraction nor the direction of (1), since we have not yet multiplied the inequality by anything):

$$\frac{(-1)^{k+1}\big(F_{k-1} - F_k r_1\big)}{(-1)^k\big(F_{k-2} - F_{k-1} r_1\big)} \;\leq\; 1. \tag{2}$$

By Lemma 2, the denominator $(-1)^k(F_{k-2} - F_{k-1}r_1)$ is strictly positive, so we may multiply both sides of (2) by it without reversing the inequality:

$$(-1)^{k+1}\big(F_{k-1} - F_k r_1\big) \;\leq\; (-1)^k\big(F_{k-2} - F_{k-1} r_1\big). \tag{3}$$

Write $(-1)^k = -(-1)^{k+1}$ on the right-hand side of (3) and expand both sides:

$$(-1)^{k+1}F_{k-1} - (-1)^{k+1}F_k r_1 \;\leq\; -(-1)^{k+1}F_{k-2} + (-1)^{k+1}F_{k-1} r_1.$$

Move every F_{k-2}- and F_{k-1}-term (the ones without r_1) to the left, and every r_1-term to the right:

$$(-1)^{k+1}F_{k-1} + (-1)^{k+1}F_{k-2} \;\leq\; (-1)^{k+1}F_{k-1}r_1 + (-1)^{k+1}F_k r_1,$$

i.e.

$$(-1)^{k+1}\big(F_{k-1} + F_{k-2}\big) \;\leq\; (-1)^{k+1}\big(F_{k-1} + F_k\big) r_1.$$

By the Fibonacci recurrence, $F_{k-1} + F_{k-2} = F_k$ and $F_{k-1} + F_k = F_{k+1}$, so this reads

$$(-1)^{k+1}F_k \;\leq\; (-1)^{k+1}F_{k+1}\, r_1.$$

Divide both sides by $F_{k+1} > 0$ (this preserves the inequality since $F_{k+1} > 0$):

$$(-1)^{k+1}\frac{F_k}{F_{k+1}} \;\leq\; (-1)^{k+1} r_1,$$

which is the assertion of the lemma. □

Lemma 4 (Cassini-type identity)**.** *For every integer $N \geq 2$,*

$$F_{N-2}F_{N+1} - F_{N-1}F_N = (-1)^N.$$

Equivalently, $(-1)^N\big(F_{N-2}F_{N+1} - F_{N-1}F_N\big) = 1.$

Proof. Define $a_n := F_{n-2}F_{n+1} - F_{n-1}F_n$ for $n \geq 2$.

Step 1: a one-term recursion $a_{n+1} = -a_n$.

Using the Fibonacci recurrence $F_{n+2} = F_{n+1} + F_n$,

$$a_{n+1} = F_{n-1}F_{n+2} - F_nF_{n+1} = F_{n-1}\big(F_{n+1} + F_n\big) - F_nF_{n+1} = F_{n-1}F_{n+1} + F_{n-1}F_n - F_nF_{n+1}.$$

Group the two terms containing F_{n+1}:

$$a_{n+1} = F_{n+1}\big(F_{n-1} - F_n\big) + F_{n-1}F_n.$$

Since $F_n = F_{n-1} + F_{n-2}$, we have $F_{n-1} - F_n = -F_{n-2}$, so

$$a_{n+1} = -F_{n+1}F_{n-2} + F_{n-1}F_n = -\big(F_{n-2}F_{n+1} - F_{n-1}F_n\big) = -a_n.$$

Step 2: base case. Using $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3$,

$$a_2 = F_0F_3 - F_1F_2 = (1)(3) - (1)(2) = 1.$$

Step 3: solve the recursion. From $a_{n+1} = -a_n$ we get, by induction on $n \geq 2$, $a_n = (-1)^{n-2}a_2$. Since $(-1)^{n-2} = (-1)^n$ (they differ by the even power $(-1)^{-2} = 1$), this is

$$a_n = (-1)^n a_2 = (-1)^n \cdot 1 = (-1)^n.$$

Taking $n = N$ gives $a_N = F_{N-2}F_{N+1} - F_{N-1}F_N = (-1)^N$, which is the lemma. Multiplying both sides by $(-1)^N$ (and using $(-1)^{2N} = 1$) gives the equivalent form $(-1)^N(F_{N-2}F_{N+1} - F_{N-1}F_N) = 1$. □

Remark. *This is a close relative of* Cassini's identity $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$*; both are instances of the general identity* $F_mF_{n+1} - F_{m+1}F_n = (-1)^nF_{m-n}$ *(a d'Ocagne-type identity), here with* $m = N-2,\ n = N-1$ *... but the direct two-line recursion argument above is self-contained and needs nothing beyond the Fibonacci recurrence.*

4 The main theorem

Theorem 1. *Let* $N \geq 2$ *and let* $r_1, \ldots, r_N$ *satisfy*

$$r_{k+1} = \frac{1}{r_k} - 1, \quad k = 1, \ldots, N-1, \qquad\qquad r_k \geq \frac{1}{2}, \quad k = 1, \ldots, N.$$

Then

$$r_1 r_2 \cdots r_N \;\geq\; \frac{1}{F_{N+1}}.$$

Moreover, equality holds if and only if

$$r_1 = \frac{F_N}{F_{N+1}},$$

and this single condition on r_1 *determines* $r_2, \ldots, r_N$ *uniquely (via the recursion), namely* $r_k = F_{N-k+1}/F_{N-k+2}$ *for* $k = 1, \ldots, N$*. In other words, the values* $r_1, \ldots, r_N$ *used by the Fibonacci search method form the* unique *minimizer.*

Proof. **Step 1: a telescoping product.** For each $k = 2, \ldots, N$ set

$$D_k := F_{k-2} - F_{k-1} r_1.$$

Lemma 1 says exactly that $r_k = -D_k/D_{k-1}$ for $k = 2, \ldots, N$ (where $D_1 := F_{-1} - F_0 r_1 = 0 - r_1 = -r_1$, consistent with the formula at $k = 1$ if one extends the definition of D_k down to $k = 1$). Multiplying these relations for $k = 2, \ldots, N$ telescopes:

$$r_2 r_3 \cdots r_N = \prod_{k=2}^{N} \left(-\frac{D_k}{D_{k-1}} \right) = (-1)^{N-1} \cdot \frac{D_2}{D_1} \cdot \frac{D_3}{D_2} \cdots \frac{D_N}{D_{N-1}} = (-1)^{N-1} \frac{D_N}{D_1},$$

because every D_k with $2 \le k \le N-1$ appears once in a numerator and once in a denominator and cancels. Substituting $D_1 = -r_1$,

$$r_2 \cdots r_N = (-1)^{N-1} \frac{D_N}{-r_1} = (-1)^N \frac{D_N}{r_1}.$$

Multiplying by r_1 (this is legitimate whether or not r_1 is being solved for, since r_1 just cancels):

$$r_1 r_2 \cdots r_N = (-1)^N D_N = (-1)^N \big(F_{N-2} - F_{N-1} r_1 \big). \tag{4}$$

This is the identity that the source text states with the words "by substituting expressions for $r_1, \ldots, r_N$ from Lemma 7.1 and performing the appropriate cancellations"; (4) makes those cancellations explicit.

Step 2: expand and bound. Expand the right side of (4):

$$r_1 \cdots r_N = (-1)^N F_{N-2} + (-1)^{N+1} F_{N-1}\, r_1,$$

using $-(-1)^N = (-1)^{N+1}$. Apply Lemma 3 with $k = N$ (legitimate, $N \ge 2$):

$$(-1)^{N+1} r_1 \;\ge\; (-1)^{N+1} \frac{F_N}{F_{N+1}}.$$

Since $F_{N-1} \ge 0$ (indeed $F_{N-1} > 0$ for $N \ge 2$), multiplying this inequality by F_{N-1} preserves its direction:

$$F_{N-1}(-1)^{N+1} r_1 \;\ge\; F_{N-1}(-1)^{N+1} \frac{F_N}{F_{N+1}}.$$

Adding $(-1)^N F_{N-2}$ to both sides,

$$r_1 \cdots r_N = (-1)^N F_{N-2} + (-1)^{N+1} F_{N-1} r_1 \;\ge\; (-1)^N F_{N-2} + (-1)^{N+1} \frac{F_{N-1} F_N}{F_{N+1}}.$$

Factor out $(-1)^N$ on the right (using $(-1)^{N+1} = -(-1)^N$):

$$r_1 \cdots r_N \;\ge\; (-1)^N \left(F_{N-2} - \frac{F_{N-1} F_N}{F_{N+1}} \right) = (-1)^N \cdot \frac{F_{N-2} F_{N+1} - F_{N-1} F_N}{F_{N+1}}.$$

By Lemma 4, the numerator satisfies $(-1)^N(F_{N-2}F_{N+1} - F_{N-1}F_N) = 1$, so

$$r_1 \cdots r_N \;\geq\; \frac{1}{F_{N+1}},$$

which is the desired inequality.

Step 3: equality case. Every inequality used above traces back to a single step: multiplying the inequality of Lemma 3 by the nonnegative constant F_{N-1}. Two sub-cases:

- If $N \geq 3$, then $F_{N-1} > 0$ strictly, so multiplying by it preserves *strict* equality/inequality status: equality holds throughout Step 2 if and only if Lemma 3's inequality is itself an equality, i.e.
$$(-1)^{N+1} r_1 = (-1)^{N+1} \frac{F_N}{F_{N+1}} \iff r_1 = \frac{F_N}{F_{N+1}}.$$
- If $N = 2$, then $F_{N-1} = F_1 = 1 > 0$ as well, so the same conclusion holds; there is no genuinely degenerate case in this range since $F_{N-1} > 0$ for all $N \geq 2$ under our indexing convention.

Hence in every case $N \geq 2$,

$$r_1 \cdots r_N = \frac{1}{F_{N+1}} \iff r_1 = \frac{F_N}{F_{N+1}}.$$

Step 4: this pins down the whole sequence. The recursion $r_{k+1} = 1/r_k - 1$ is a deterministic first-order recursion: once r_1 is fixed, $r_2, r_3, \ldots, r_N$ are determined one at a time. So $r_1 = F_N/F_{N+1}$ determines a unique sequence $r_1, \ldots, r_N$, and by Lemma 1 (used to *compute*, not merely bound) this sequence is

$$r_k = -\frac{F_{k-2} - F_{k-1} \cdot F_N/F_{N+1}}{F_{k-3} - F_{k-2} \cdot F_N/F_{N+1}} = -\frac{F_{k-2}F_{N+1} - F_{k-1}F_N}{F_{k-3}F_{N+1} - F_{k-2}F_N} \qquad (k \geq 2),$$

which one checks (repeated use of the Fibonacci recurrence, or a short separate induction — this is Exercise 7.4 in the source text) simplifies to the closed form

$$r_k = \frac{F_{N-k+1}}{F_{N-k+2}}, \qquad k = 1, \ldots, N.$$

Indeed this closed form manifestly satisfies $r_{k+1} = 1/r_k - 1$:

$$\frac{1}{r_k} - 1 = \frac{F_{N-k+2}}{F_{N-k+1}} - 1 = \frac{F_{N-k+2} - F_{N-k+1}}{F_{N-k+1}} = \frac{F_{N-k}}{F_{N-k+1}} = r_{k+1},$$

using $F_{N-k+2} - F_{N-k+1} = F_{N-k}$ (Fibonacci recurrence rearranged), and it satisfies $r_1 = F_N/F_{N+1}$ at $k = 1$, matching the value forced in Step 3. $\square$

Remark (Translation back to the Fibonacci search method)**.** *Recall $\rho_k = 1 - r_k$. Theorem 1 says the minimum value of the overall reduction factor $(1-\rho_1)\cdots(1-\rho_N)$ over all feasible sequences is exactly $1/F_{N+1}$, attained uniquely by*

$$\rho_k = 1 - r_k = 1 - \frac{F_{N-k+1}}{F_{N-k+2}}, \qquad k = 1, \dots, N,$$

which is precisely the sequence of ratios $\rho_k = 1 - F_{N-k+1}/F_{N-k+2}$ used to define the Fibonacci search method. This is the announced conclusion: the Fibonacci method's choices are the only *feasible choices that attain the best possible worst-case reduction factor $1/F_{N+1}$ after N function evaluations.*

Summary of the logical chain

Prop. 1 ($\rho \leftrightarrow r$)	$\Longrightarrow$	problem restated in r's
Lemma 1 (closed form for r_k)	$\Longrightarrow$	Lemma 2 (sign lemma) $\Longrightarrow$ Lemma 3 (bound on r_1)
Lemma 1 + telescoping	$\Longrightarrow$	$r_1 \cdots r_N = (-1)^N (F_{N-2} - F_{N-1} r_1)$ (4)
(4) + Lemma 3 + Lemma 4 (Cassini-type identity)	$\Longrightarrow$	$r_1 \cdots r_N \geq 1/F_{N+1}$, with equality iff $r_1 = F_N/F_{N+1}$.