SRI SATHYA SAI INSTITUTE OF HIGHER LEARNING

(Deemed to be University)

Department of Mathematics and Computer Science

TUTORIAL-II SOLUTIONS

Topics: Continuous Probability Distributions

UDSC-201: Probability and Statistics	Instructor: Dr. D Bhanu Prakash	Date: 19-08-2026

Question 1

[BTL-3: Apply — compute k, CDF, probability, mean and variance from a given pdf]

A continuous random variable X has probability density function

$$f(x) = k\,x,\ \ 0 \le x \le 4, \quad \text{and } f(x) = 0 \text{ otherwise.}$$

(a) Find the value of k. (b) Find the CDF F(x). (c) Find $P(1 \le X \le 3)$. (d) Find the mean and variance of X.

Solution

(a) Finding k

Since f is a valid pdf, the total area under the curve must equal 1:

$$\int_0^4 k\,x \ \ dx = 1$$

$$k \times [\ x^2/2\ ]_0^4 = 1$$

$$k \times 8 = 1 \quad \Longrightarrow \quad k = 1/8$$

(b) CDF

For $0 \le x \le 4$:

$$F(x) = \int_0^x (t/8)\,dt = x^2/16$$

Hence:

$$F(x) = 0,\ x < 0; \quad F(x) = x^2/16,\ 0 \le x \le 4; \quad F(x) = 1,\ x > 4$$

(c) $P(1 \le X \le 3)$

$$P(1 \le X \le 3) = F(3) - F(1) = 9/16 - 1/16 = 8/16 = 1/2$$

(d) Mean and Variance

$$E(X) = \int_0^4 x \cdot (x/8)\,dx = 64/24 = 8/3$$

$$E(X^2) = \int_0^4 x^2 \cdot (x/8)\,dx = 256/32 = 8$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 8 - (8/3)^2 = 8 - 64/9 = 8/9$$

Answer: k = 1/8; $F(x) = x^2/16$ on [0,4]; $P(1 \le X \le 3) = 1/2$; E(X) = 8/3; Var(X) = 8/9

Question 2

[BTL-3: Apply — Uniform distribution probability and moments]

A bus arrives at a stop at a uniformly random time between 9:00 AM and 9:20 AM. Let X (in minutes after 9:00) denote the arrival time, so X ~ Uniform(0, 20). Find (a) the pdf of X, (b) $P(5 < X < 15)$, (c) the mean and variance, and (d) the probability that a passenger who arrives at 9:08 waits more than 5 minutes.

Solution

(a) pdf

$$f(x) = 1/(b-a) = 1/20, \quad 0 \le x \le 20$$

(b) Probability

$$P(5 < X < 15) = (15 - 5)/20 = 10/20 = 0.5$$

(c) Mean and variance

$$E(X) = (a + b)/2 = (0 + 20)/2 = 10$$

$$\mathrm{Var}(X) = (b - a)^2/12 = 400/12 = 33.33$$

(d) Wait more than 5 minutes after 9:08

Passenger arrives at t = 8. Waiting more than 5 minutes means the bus arrives after t = 13:

$$P(X > 13) = (20 - 13)/20 = 7/20 = 0.35$$

Answer: f(x)=1/20 on [0,20]; P(5<X<15)=0.5; E(X)=10 min; Var(X)=33.33 min²; P(wait>5 min)=0.35

Question 3

[BTL-3: Apply — Exponential distribution and the memoryless property]

The lifetime of an electronic component (in years) follows an Exponential distribution with mean life 5 years. (a) Find the rate parameter λ and the pdf. (b) Find the probability the component survives more than 8 years. (c) Given that it has already survived 4 years, find the probability it survives 4 more years, and comment on what property this illustrates.

Solution

(a) Parameter and pdf

Since the mean of an Exponential(λ) variable is 1/λ:

$$1/\lambda = 5 \quad \Longrightarrow \quad \lambda = 0.2$$

$$f(x) = \lambda\, e^{-\lambda x} = 0.2\, e^{-0.2x}, \quad x \ge 0$$

(b) P(X > 8)

The reliability (survival) function of the Exponential distribution is:

$$R(x) = P(X > x) = e^{-\lambda x}$$

$$P(X > 8) = e^{-0.2(8)} = e^{-1.6} \approx 0.2019$$

(c) Memoryless property

For the Exponential distribution, for any s, t > 0:

$$P(X > s + t \mid X > s) = P(X > t)$$

$$P(X > 8 \mid X > 4) = P(X > 4) = e^{-0.2(4)} = e^{-0.8} \approx 0.4493$$

This confirms the memoryless property: the additional 4-year survival probability does not depend on the 4 years already survived — the component does not 'age' in a probabilistic sense.

Answer: $\lambda = 0.2$/yr; $P(X>8) \approx 0.2019$; $P(X>8|X>4) = P(X>4) \approx 0.4493$ (memoryless)

Question 4

[BTL-3: Apply — Normal distribution, standardization (Z-scores)]

Scores on a standardized test are normally distributed with mean $\mu = 500$ and standard deviation $\sigma = 100$. (a) Find the probability that a randomly chosen student scores between 450 and 650. (b) Find the minimum score needed to be in the top 10% of students.

Solution

(a) $P(450 < X < 650)$

Standardize using the transformation:

$$Z = (X - \mu)/\sigma$$

$$Z_1 = (450-500)/100 = -0.5 \qquad Z_2 = (650-500)/100 = 1.5$$

$$P(450 < X < 650) = \Phi(1.5) - \Phi(-0.5)$$

$$= 0.9332 - 0.3085 = 0.6247$$

(b) Top 10% cutoff

We need x such that $P(X > x) = 0.10$, i.e. $\Phi(z) = 0.90$, which from standard Normal tables gives $z \approx 1.2816$:

$$x = \mu + z\sigma = 500 + 1.2816 \times 100 \approx 628.16$$

Answer: $P(450<X<650) \approx 0.6247$; top-10% cutoff score ≈ 628.2

Question 5

[BTL-4: Analyze — justify and apply the Normal approximation to the Binomial with continuity correction]

A fair coin is tossed $n = 400$ times. Let X be the number of heads. (a) Justify whether a Normal approximation is appropriate. (b) Using the Normal approximation with continuity correction, find $P(X \geq 210)$. (c) Compare briefly what would go wrong if continuity correction were omitted.

Solution

(a) Justification for the approximation

Here $X \sim \text{Binomial}(n = 400, p = 0.5)$. The rule of thumb for a good Normal approximation is that both np and $n(1-p)$ should be reasonably large (commonly ≥ 5, ideally ≥ 10):

$$n\,p = 400(0.5) = 200 \geq 10 \qquad n(1-p) = 400(0.5) = 200 \geq 10$$

Both conditions are comfortably satisfied, so the Normal approximation is justified.

(b) Parameters and continuity-corrected probability

$$\mu = n\,p = 200 \qquad \sigma = \sqrt{[n\,p(1-p)]} = \sqrt{100} = 10$$

Since X is discrete and we approximate with a continuous Normal variable, apply the continuity correction, replacing $X \geq 210$ with $X > 209.5$:

$$P(X \geq 210) \approx P(Y > 209.5), \quad Y \sim \mathrm{N}(200, 10^2)$$

$$Z = (209.5 - 200)/10 = 0.95$$

$$P(Y > 209.5) = 1 - \Phi(0.95) = 1 - 0.8289 = 0.1711$$

(c) Effect of omitting the continuity correction

Without the correction we would compute $Z = (210-200)/10 = 1.0$, giving $P \approx 1 - \Phi(1.0) = 0.1587$. This systematically underestimates $P(X \geq 210)$ here because the discrete boundary at $X = 210$ is being treated as though probability mass exactly at 210 does not count; the 0.5 correction accounts for that boundary cell and makes the continuous approximation align with the discrete distribution.

Answer: Approximation justified ($np, n(1-p) \geq 10$); $P(X\geq 210) \approx 0.1711$ with correction vs. 0.1587 without

Question 6

[BTL-3: Apply — Erlang distribution as a sum of Exponential stages]

Customers arrive at a service counter according to a Poisson process at rate $\lambda = 4$ per hour. Let T be the time until the 3rd customer arrives, so $T \sim \text{Erlang}(k = 3, \lambda = 4)$. (a) Write the pdf of T. (b) Find E(T) and Var(T). (c) Find the probability that the 3rd customer arrives within 30 minutes (0.5 hour).

Solution

(a) pdf of the Erlang(k, λ) distribution

$$f(t) = \lambda^k\, t^{k-1}\, e^{-\lambda t} / (k-1)!, \quad t \geq 0$$

Substituting $k = 3$, $\lambda = 4$:

$$f(t) = 4^3\, t^2\, e^{-4t} / 2! = 32\, t^2\, e^{-4t}, \quad t \geq 0$$

(b) Mean and variance

$$E(T) = k/\lambda = 3/4 = 0.75 \text{ hours}$$

$$\text{Var}(T) = k/\lambda^2 = 3/16 = 0.1875 \text{ hours}^2$$

(c) P(T ≤ 0.5)

For integer k, the Erlang CDF can be written using the Poisson relationship: $T \leq t$ is equivalent to at least k Poisson(λt) arrivals occurring by time t:

$$P(T \leq t) = 1 - \Sigma_{n=0}^{k-1}\, (\lambda t)^n\, e^{-\lambda t} / n!$$

With $\lambda t = 4(0.5) = 2$ and $k = 3$:

$$P(T \leq 0.5) = 1 - e^{-2}\, [1 + 2 + 2^2/2!] = 1 - e^{-2}(5)$$

$$= 1 - 0.1353 \times 5 = 1 - 0.6767 = 0.3233$$

Answer: $f(t)=32t^2e^{-4t}$; $E(T)=0.75$ hr; $\text{Var}(T)=0.1875$ hr²; $P(T\leq 0.5) \approx 0.3233$

Question 7

[BTL-4: Analyze — recognize Gamma model parameters and relate to component distributions]

The total repair time (in hours) for a machine is modeled as $X \sim \text{Gamma}(\text{shape } \alpha = 2.5, \text{ rate } \beta = 0.5)$. (a) Write the pdf of X. (b) Find the mean and variance. (c) Explain how this Gamma model relates to the Exponential and Erlang distributions, and state the condition under which it reduces to each.

Solution

(a) pdf of the Gamma(α, β) distribution

$$f(x) = \beta^\alpha\, x^{\alpha-1}\, e^{-\beta x} / \Gamma(\alpha), \quad x \geq 0$$

With $\alpha = 2.5$, $\beta = 0.5$:

$$f(x) = 0.5^{2.5}\, x^{1.5}\, e^{-0.5x} / \Gamma(2.5), \quad x \geq 0$$

(b) Mean and variance

$$E(X) = \alpha/\beta = 2.5/0.5 = 5 \text{ hours}$$

$$\mathrm{Var}(X) = \alpha/\beta^2 = 2.5/0.25 = 10 \text{ hours}^2$$

(c) Relation to Exponential and Erlang

The Gamma family is the general 'waiting-time' distribution for a continuous shape parameter α > 0:

$$\alpha = 1 \quad \Rightarrow \quad \text{Gamma}(1, \beta) \equiv \text{Exponential}(\beta)$$

$$\alpha = k \in \mathbb{Z}^+ \quad \Rightarrow \quad \text{Gamma}(k, \beta) \equiv \text{Erlang}(k, \beta)$$

So Exponential is the special Gamma case of a single stage (α = 1), Erlang is the special case of an integer number of stages (α = k), and the Gamma distribution generalizes both to any positive real shape parameter, which is why here α = 2.5 (non-integer) can still be handled by Gamma but not by Erlang directly.

Answer: E(X)=5 hr, Var(X)=10 hr²; Gamma generalizes Exponential (α=1) and Erlang (α=k integer)

Question 8

[BTL-4: Analyze — interpret Weibull shape parameter and compute reliability/failure rate behavior]

The lifetime (in years) of a mechanical part follows a Weibull distribution with shape parameter β = 2 and scale parameter η = 4. (a) Write the pdf and reliability function. (b) Find the probability the part survives beyond 5 years. (c) Determine whether the failure rate is increasing, constant, or decreasing with age, and justify using the shape parameter.

Solution

(a) pdf and reliability function

$$f(x) = (\beta/\eta)\, (x/\eta)^{\beta-1}\, e^{-(x/\eta)^\wedge\beta}, \quad x \geq 0$$

$$R(x) = P(X > x) = e^{-(x/\eta)^\wedge\beta}$$

(b) P(X > 5)

$$R(5) = e^{-(5/4)^\wedge 2} = e^{-1.5625} \approx 0.2096$$

(c) Failure-rate behavior

The Weibull hazard (instantaneous failure) rate function is:

$$h(x) = (\beta/\eta)\, (x/\eta)^{\beta-1}$$

Since β = 2 > 1, h(x) is an increasing function of x. This means the part exhibits wear-out behavior: it becomes more likely to fail as it ages (unlike β = 1, which reduces to a constant-rate Exponential model, or β < 1, which would indicate a decreasing/'infant mortality' failure rate).

Answer: R(5) ≈ 0.2096; β=2>1 ⇒ increasing failure rate (wear-out behavior)

Question 9

[BTL-4: Analyze — apply the Lognormal model via the underlying Normal transform]

The size of insurance claims X (in thousand $) is modeled as Lognormal, meaning ln(X) ~ N(μ = 1.5, σ² = 0.36). (a) Find the mean and variance of X itself (not ln X). (b) Find the probability a claim exceeds $10,000. (c) Explain why the Normal model is applied to ln(X) rather than to X directly.

Solution

(a) Mean and variance of the Lognormal X

If $\ln(X) \sim N(\mu, \sigma^2)$, then:

$$E(X) = e^{\mu + \sigma^2/2}$$

$$\mathrm{Var}(X) = (e^{\sigma^2} - 1)\, e^{2\mu + \sigma^2}$$

With $\mu = 1.5$, $\sigma^2 = 0.36$:

$$E(X) = e^{1.5 + 0.18} = e^{1.68} \approx 5.365 \quad \text{(i.e., \$5,365)}$$

$$\mathrm{Var}(X) = (e^{0.36} - 1)\, e^{3.18} \approx (1.4333 - 1)(24.03) \approx 10.42$$

(b) P(X > 10)

Take natural logs of both sides and standardize using $\sigma = \sqrt{0.36} = 0.6$:

$$P(X > 10) = P(\ln X > \ln 10) = P(Z > (\ln 10 - 1.5)/0.6)$$

$$\ln 10 \approx 2.3026 \quad \Rightarrow \quad Z = (2.3026 - 1.5)/0.6 \approx 1.338$$

$$P(X > 10) = 1 - \Phi(1.338) \approx 1 - 0.9096 = 0.0904$$

(c) Why model ln(X) rather than X

Claim sizes are strictly positive and typically right-skewed (many small claims, a few very large ones). Taking logs compresses large values and symmetrizes the distribution, so that ln(X) is well approximated by a symmetric Normal distribution — X itself cannot be Normal since Normal variables can be negative, which claim amounts cannot.

Answer: $E(X) \approx \$5{,}365$; $\mathrm{Var}(X) \approx 10.42$ (thousand \$)²; $P(X>10) \approx 0.0904$

Question 10

[BTL-4: Analyze — apply the Beta distribution to a bounded proportion and interpret its shape]

The proportion X of defective items in a large shipment (a value always between 0 and 1) is modeled with a Beta distribution with parameters $\alpha = 2$, $\beta = 5$. (a) Write the pdf of X. (b) Find the mean and variance. (c) Determine the mode of the distribution and interpret what it says about typical defect proportions. (d) Explain why Beta (rather than Normal) is the natural choice here.

Solution

(a) pdf of the Beta(α, β) distribution

$$f(x) = [1/B(\alpha, \beta)]\, x^{\alpha-1} (1 - x)^{\beta-1}, \quad 0 \le x \le 1$$

With $\alpha = 2$, $\beta = 5$, and $B(2,5) = \Gamma(2)\Gamma(5)/\Gamma(7) = 1!\cdot 4!/6! = 24/720 = 1/30$:

$$f(x) = 30\, x\, (1 - x)^4, \quad 0 \le x \le 1$$

(b) Mean and variance

$$E(X) = \alpha/(\alpha + \beta) = 2/7 \approx 0.2857$$

$$\mathrm{Var}(X) = \alpha\beta \,/\, [(\alpha + \beta)^2 (\alpha + \beta + 1)] = 10/(49 \times 8) = 10/392 \approx 0.0255$$

(c) Mode

For $\alpha, \beta > 1$, the mode of the Beta distribution is:

$$\text{Mode} = (\alpha - 1)/(\alpha + \beta - 2) = 1/5 = 0.2$$

The most likely defect proportion is 20%, somewhat lower than the mean of 28.6% — consistent with the distribution's right skew (β = 5 > α = 2 pulls probability mass toward smaller x).

(d) Why Beta rather than Normal

X is a proportion and is restricted to the fixed interval [0, 1]. The Normal distribution has support over all real numbers and would assign positive probability to impossible values ($X < 0$ or $X > 1$). The Beta distribution is defined exactly on [0, 1] and its two shape parameters flexibly capture skewed, bounded behavior, making it the natural conjugate model for proportions and probabilities.

Answer: $f(x)=30x(1-x)^4$; $E(X)\approx 0.2857$; $Var(X)\approx 0.0255$; Mode=0.2; Beta chosen for its [0,1] support