From Dice to Distributions

A Guided Tour of Discrete Random Variables

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Instructor:	Dr. D Bhanu Prakash
Instructor Email:	dbhanuprakash233@gmail.com
Department:	Department of Mathematics and Computer Science
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Overview: The Setup Question Behind Each Distribution

Every distribution below is illustrated using rolls of a single fair six-sided die. The table summarizes the core question each distribution answers, in increasing order of complexity.

Distribution	**Setup Question**
Uniform	What face shows up on a single roll of the die?
Binomial	In n rolls, how many times does a specific face (e.g., a 6) appear?
Poisson	As rolls happen continuously at a steady rate, how many 6s occur in a fixed time window?
Geometric	How many rolls are needed until the *first* 6 appears?
Negative Binomial	How many rolls are needed until the r-*th* 6 appears?
Hypergeometric	If dice are drawn *without replacement* from a fixed mixed batch, how many "success" dice are drawn?

1 Discrete Uniform — The Baseline

Setup: One roll of a fair die. X = the face that shows up.

Every outcome is equally likely — that is the entire idea of "uniform."

$$P(X = k) = \frac{1}{6}, \quad k = 1, 2, 3, 4, 5, 6$$

- **Mean:** $E[X] = \dfrac{1 + 2 + \cdots + 6}{6} = 3.5$
- **Variance:** $\mathrm{Var}(X) = \dfrac{n^2 - 1}{12} = \dfrac{35}{12} \approx 2.92$

This is the "atom" distribution — it describes a single roll with no notion of success/failure, counting, or repetition. Every other distribution is built by adding one more layer of structure on top of repeated die rolls.

2 Binomial — Counting Successes in a Fixed Number of Rolls

Setup: Redefine "success" = rolling a 6 (probability $p = 1/6$), "failure" = anything else ($q = 5/6$). Roll the die n times. X = number of 6s obtained.

$$P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots, n$$

Example: Roll the die 10 times. Probability of exactly 3 sixes:

$$P(X = 3) = \binom{10}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7 \approx 0.155$$

- **Mean:** np
- **Variance:** npq

Key conditions: fixed number of trials n, constant probability p each trial, independence, and trials with replacement (each roll is a fresh die roll — the die "resets").

3 Poisson — Counting Successes When Trials Become Dense and Rare

Setup: Instead of a *fixed number* of rolls, imagine rolling the die continuously at a steady rate — say once every second — and asking how many 6s occur in a fixed time window, like one hour.

Poisson emerges as the limiting case of Binomial when $n \to \infty$ and $p \to 0$ such that $np \to \lambda$ stays fixed.

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Example: Rolling once every second for an hour gives $n = 3600$ rolls, so $\lambda = 3600 \times \frac{1}{6} = 600$. Poisson approximates the number of 6s in that hour — the approximation is good precisely because n is huge.

- **Mean:** λ
- **Variance:** λ (mean = variance is the signature quirk of Poisson)

The conceptual jump: instead of tracking a fixed, small n, Poisson describes counts of rare events over a continuum (time, space, or a very large number of trials), decoupling the "count" from any specific trial size.

4 Geometric — Waiting for the First Success

Setup: Keep rolling the die until the *first* 6 appears. X = number of rolls needed (including the successful one).

$$P(X = k) = q^{k-1} p = \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right), \quad k = 1, 2, 3, \ldots$$

Example: Probability that the first 6 appears exactly on the 4th roll:

$$P(X=4) = \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right) \approx 0.0965$$

- **Mean:** $\frac{1}{p} = 6$ (on average, 6 rolls are needed to see a 6)
- **Variance:** $\frac{q}{p^2} = \frac{5/6}{1/36} = 30$

Geometric has the famous **memoryless property**: if 5 rolls have already passed without a 6, the probability of needing exactly 3 more rolls is the same as starting fresh. The die "does not remember" past bad luck.

5 Negative Binomial — Waiting for the r-th Success

Setup: Generalize Geometric: keep rolling until the r**-th** 6 appears. X = number of rolls needed to see the r-th 6.

$$P(X=k) = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, r+2, \ldots$$

Example: Probability that the 3rd six occurs exactly on the 8th roll ($r = 3$, $k = 8$):

$$P(X=8) = \binom{7}{2} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5 \approx 0.0651$$

- **Mean:** $\frac{r}{p}$ — e.g., for $r = 3$, expect 18 rolls
- **Variance:** $\frac{rq}{p^2}$

Notice the layering: Binomial fixes n and asks about successes; Negative Binomial fixes the *number of successes* (r) and asks about the number of trials — it is Binomial's "mirror image."

6 Hypergeometric — Sampling Without Replacement

Setup: Imagine a bag of 20 dice: 6 are red and 14 are blue. Draw 5 dice from the bag **without replacement** and let X = number of red dice drawn.

The critical new complexity: unlike every distribution above, trials are **not independent**, and probabilities change with each draw because the finite population is being depleted.

$$P(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

where $N = 20$ (total dice), $K = 6$ (red dice), $n = 5$ (draws), and k = red dice drawn.

Example: Probability of drawing exactly 2 red dice in 5 draws:

$$P(X=2) = \frac{\binom{6}{2}\binom{14}{3}}{\binom{20}{5}} = \frac{15 \times 364}{15504} \approx 0.352$$

- **Mean:** $n\dfrac{K}{N} = 5 \times \dfrac{6}{20} = 1.5$
- **Variance:** $n\dfrac{K}{N} \cdot \dfrac{N-K}{N} \cdot \dfrac{N-n}{N-1}$

The extra finite-population correction factor $\dfrac{N-n}{N-1}$ is unique to Hypergeometric, and is exactly why it is the most complex distribution on this list. If sampling were done *with* replacement instead, this would collapse back into Binomial — **Hypergeometric is Binomial's more complicated cousin, adjusted for a finite, shrinking population.**

The Big Picture

Distribution	What Varies	What's Fixed	Independent?
Uniform	single outcome	—	—
Binomial	# successes	# trials (n)	Yes
Poisson	# rare events	rate (λ) over continuum	Yes
Geometric	# trials	1 success needed	Yes
Negative Binomial	# trials	r successes needed	Yes
Hypergeometric	# successes	finite population, no replacement	**No**

The complexity ladder tracks: *counting a single roll → counting successes among fixed trials → counting rare events at scale → counting trials until success(es) → counting successes when trials interfere with each other because the population is finite.*